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Numerical Solution of Korteweg De-Vries Equation Using Method Of Lines

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Abstract

The Korteweg-de Vries (KdV) equation was initially formulated to investigate the propagation of long waves in shallow water. It describes a phenomenon in which taller waves eventually catch up to and pass shorter waves, leading to a nonlinear interaction characterized by a phase change, while preserving the form and speed of the waves. In this study, we utilize the third-order KdV equation to simulate water waves with surface tension. The Method of Lines (MOL) is employed as a numerical approach to solve the KdV equation, converting the partial differential equation into a system of ordinary differential equations (ODEs). Specifically, we utilize the fourth-order Runge-Kutta method within the MOL framework. The primary objective of this research is to solve the KdV equation using the Method of Lines with the Runge-Kutta 4th Order method and assess its accuracy by comparing the numerical results with the analytical solution. MATLAB softwa This research contributes to the understanding and application of the Method of Lines for solving the KdV equation, offering insights into simulating water wave phenomena. The findings highlight the accuracy and reliability of the proposed numerical approach, providing a valuable tool for studying nonlinear wave propagation in various scientific and engineering domains.

Keywords Korteweg De-Vries equation; method of lines; Runge- Kutta 4th Order method; analytical and numerical.

Introduction

The purpose of KdV equation originally is to examine the long wave propagation in shallow water as been first introduced in an article written by Dutch mathematicians Korteweg and De Vries. Zabuskay and Kruskal showed that the wave solutions survived after interactions and were referred to as "solitons" in their investigation on the KdV equation. In the study of nonlinear wave processes, the soliton idea is crucial. Physically, the taller (faster) wave is on the left of the shorter (slower) wave when two solitons of different amplitudes (and thus, of different speeds) are placed far apart on the actual line. The taller wave ultimately catches up to the shorter one and then passes it. According to the KdV equation, when this occurs, they experience a nonlinear interaction and come out of it fully undamaged in terms of form and speed, with only a phase change. Recently, the method of lines has been introduced to calculate the small time in the KdV equation

The Method of lines(MOL) is one method for solving PDEs, and as a result, it is of broad interest in science and engineering (Schiesser, 1991). MOL generally is a step for solutions towards time PDE with auxiliary conditions.The numerical treatment of partial differential equations (PDEs) representing nonlinear wave phenomena and specific varieties of solitary waves has attracted a lot of attention in recent years. The third-order Korteweg-de Vries (KdV) equation that can be used to simulate water waves with surface tension is the main study of this work.

The objective of this study is to compute a numerical solution for the KdV equation using the method of lines (MOL) and compare it with the KdV equation's analytical solution. The alternative is by showing how KdV equation solitons moving at various speeds can merge and appear in a second numerical solution.

Material and methods

Korteweg-de Vries (KdV) equation

A famous nonlinear partial differential equation (PDE) that was first developed to simulate shallow water flow is the Korteweg-de Vries equation (KdV). In addition to its use in hydrodynamics, the KdV has been investigated to shed light on several intriguing mathematical characteristics. To create solitons—traveling waves that don't vary in form or speed—the KdV, in particular, strikes a compromise between front sharpening and dispersion. The shallow-water waves, bubble liquid mixes, and one-dimensional waves of tiny but limited amplitude in dispersive systems can be described by the third-order Korteweg-de Vries (KdV) equation. Third order Korteweg-de Vries (KdV) equation is given by,

$$\frac{u}{t} + u\,{}_{x} + \frac{u}{x^3} = 0 \quad \frac{{}^3u}{}$$

where,

u = wave function

x = variable respective to space

t = variable respective to time

= wave speed

= coefficient related to dispersion

The equation that will be used is $_t + _x + 4.84 \times 10^{-4}{}_{xxx} = 0$ where initial condition is $(x, 0) = (x) = 0.9eh^2\ 12.45\{x - 6\ \}$

Method Of Lines

The method of lines, or MOL for short is widely regarded as a thorough and effective method for solving time-dependent PDEs numerically. Approximating the spatial derivatives is the first stage in this procedure, and the resulting system of semi-discrete ordinary differential equations (ODEs) is then integrated in time. As a result, the technique of lines comes close to solving PDEs using integrators of ODEs.

Runge-Kutta 4th Order Method

The most used Runge-Kutta technique is the Runge-Kutta 4th Order method (RK4). One of the way to solve the method of lines is by RK4. the general equation for RK4 will be obtained

$$_{+1} = \ + \frac{1}{6}(_1 + 2_2 + 2_3 + _4)$$

where

$$_1 = h\ x,\ (\quad)$$

$$_2 = h\ x\left(+ \frac{h}{2},\ + \frac{_1}{2}\right)$$

$$_3 = h\ x\left(+ \frac{h}{2},\ + \frac{_2}{2}\right)$$

$$_4 = h\ x\ (+h,\ + _3\ .\quad)$$

Discretization of KdV equation using MOL

Consider a KdV equation given by

$$_t + _x + 4.84 \times 10^{-4}{}_{xxx} = 0$$

Then subsidivide the spatial domain into N+1 uniformly regular meshes by the lines and denote at these grid points as

$$= h\ (\ = 0,1,2,\ldots,\ ) \qquad \text{where } h = 0.1$$

A numerical approximation of the first and third derivative of for the spatial derivative using central differencing is the third-order central difference. To summarize, the discretization of $_x$ and $_{xxx}$ of all *i* is shown as

$$_{x} = \frac{-U_{i-1} + U_{i+1}}{2h}$$

$$_{xxx} = \frac{-U_{i-2} + 2U_{i-1} - 2U_{i+1} + U_{i+2}}{2h^3}$$

Substitute $_x$ and $_{xxx}$ in respectively, thus a system of ODE depends on *t* in the form

$$\frac{}{t} = () \qquad = 1,2,\ldots, -1$$

is obtained where $f(U_i)$ is given as follows:

$$f() = -4.84 \times 10^{-4} \frac{-U_{i-2} + 2U_{i-1} - 2U_{i+1} + U_{i+2}}{2h^3} - \frac{-_{-1} + _{+1}}{2h}$$

Solving the system of ODEs using RK4

The RK4 formula to solve the system of ODE in (3.9) can be written as

$$_{+1} = + \frac{1}{6}(_1 + 2_2 + 2_3 + _4)$$

where

$$_1 = \Delta t \; ^j(x) \;, t \quad)$$

$$_2 = \Delta t \; ^j \left(x + \left(\frac{h}{2}\right) \;, t^j + \frac{1}{2}\right)$$

$$_3 = \Delta t \; ^j \left(x + \left(\frac{h}{2}\right), t^j + \frac{2}{2}\right)$$

$$_4 = \Delta t \; ^j \; (x + (h \;, t^j +)_3 \quad)$$

Then Runge Kutta 4th order method will be used to model the KdV equation and using MATLAB version 9.13.0.2126072 release R2022b to find and analyze the graph and numerical solution of the equation.

Analytical solution

lastly to compare the error and accuracy of the numerical method with the exact solution where exact solution is

$$(x,t) = 0.9eh^2 \; 1\{2.45x - 3.734t - 6 \;\}$$

Result and discussion

It should be noted that higher-order central differences become more susceptible to data noise and rounding errors, which may influence the approximation's accuracy. It's important to note that the approximation's accuracy is greatly influenced by the step size selection. Choosing an excessively small step size can result in numerical instability, while choosing an extremely large step size can result in less accurate approximations. Hence, $0 \leq t \leq 1$ where $\Delta t = 0.01$ was used to solve the equation.

Table 1: Table shows the value of all U of exact solution for $0 \leq \mathrm{x} \leq 2.0$ and $0 \leq \mathrm{t} \leq 0.1$

t x	0	0.02	0.04	0.06	0.08	0.1
0	2.21189E-05	1.90497E-05	1.64065E-05	1.41299E-05	1.21693E-05	1.04807E-05
0.1	0.000266746	0.000229737	0.000197863	0.00017041	0.000146766	0.000126403
0.2	0.003212027	0.002767018	0.00238358	0.002053216	0.001768596	0.001523396
0.3	0.037986968	0.032813016	0.028332226	0.024454705	0.021101448	0.018203223
0.4	0.366574965	0.326037899	0.288812859	0.254922711	0.224298613	0.196804779
0.5	0.855932256	0.879968607	0.894873011	0.899999222	0.895121221	0.880454129
0.6	0.171684364	0.196158558	0.223576605	0.254120872	0.287928524	0.325070358
0.7	0.015641726	0.018136317	0.021024017	0.02436514	0.028228688	0.032693412
0.8	0.001307298	0.001517745	0.001762035	0.002045601	0.002374741	0.002756759
0.9	0.00010846	0.000125934	0.000146222	0.000169778	0.000197128	0.000228884
1	8.99294E-06	1.04418E-05	1.21241E-05	1.40775E-05	1.63455E-05	1.8979E-05
1.1	7.45607E-07	8.65735E-07	1.00522E-06	1.16717E-06	1.35522E-06	1.57356E-06
1.2	6.18183E-08	7.17781E-08	8.33426E-08	9.67703E-08	1.12361E-07	1.30464E-07
1.3	5.12535E-09	5.95112E-09	6.90993E-09	8.02322E-09	9.31587E-09	1.08168E-08
1.4	4.24943E-10	4.93407E-10	5.72902E-10	6.65205E-10	7.72379E-10	8.9682E-10
1.5	3.5232E-11	4.09084E-11	4.74993E-11	5.51521E-11	6.40379E-11	7.43553E-11
1.6	2.92108E-12	3.39171E-12	3.93817E-12	4.57266E-12	5.30938E-12	6.1648E-12
1.7	2.42187E-13	2.81207E-13	3.26513E-13	3.79119E-13	4.402E-13	5.11123E-13
1.8	2.00797E-14	2.33148E-14	2.70712E-14	3.14327E-14	3.6497E-14	4.23772E-14
1.9	1.66481E-15	1.93303E-15	2.24447E-15	2.60609E-15	3.02597E-15	3.51349E-15
2	1.38029E-16	1.60268E-16	1.86089E-16	2.16071E-16	2.50883E-16	2.91303E-16

Figure 1: The graph of exact solution for $0 \leq x \leq 2.0$ and $0 \leq t \leq 0.1$

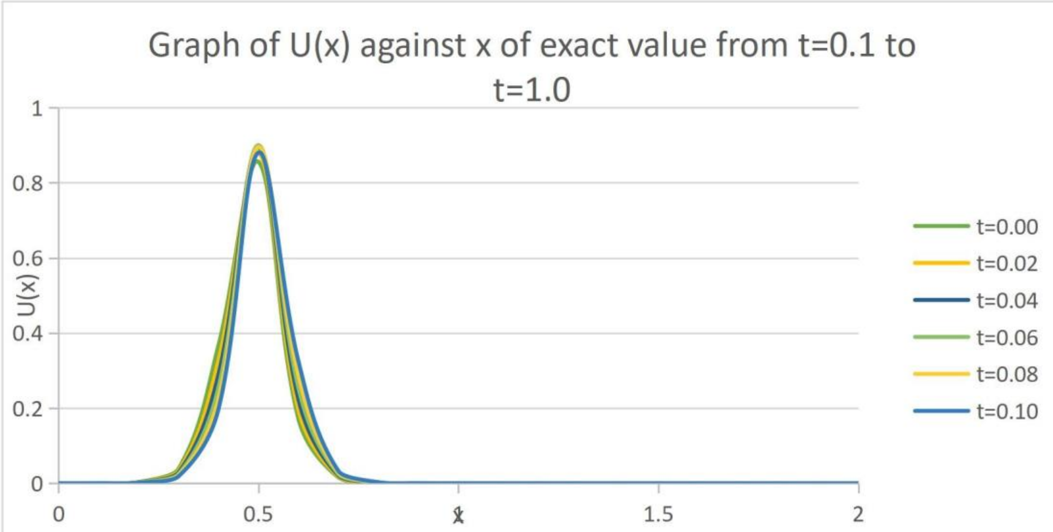


Table 2: Table shows the value of all U of numerical solution for $0 \leq x \leq 2.0$ and $0 \leq t \leq 0.1$

t x	**0.00**	**0.02**	**0.04**	**0.06**	**0.08**	**0.1**
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000
0.2	0.0032	0.0018	0.0006	0.0000	0.0000	0.0000
0.3	0.038	0.0359	0.0337	0.0315	0.0291	0.0267
0.4	0.3666	0.3444	0.3236	0.3041	0.2860	0.2692
0.5	0.8559	0.8698	0.8815	0.8910	0.8982	0.9032
0.6	0.1717	0.1801	0.1892	0.1991	0.2099	0.2216
0.7	0.0156	0.0184	0.0213	0.0241	0.0270	0.0300
0.8	0.0013	0.0020	0.0027	0.0034	0.0042	0.0050
0.9	0.0001	0.0002	0.0002	0.0003	0.0004	0.0005
1.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001
1.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Figure 2: The graph of numerical solution for $0 \leq x \leq 2.0$ and $0 \leq t \leq 0.1$

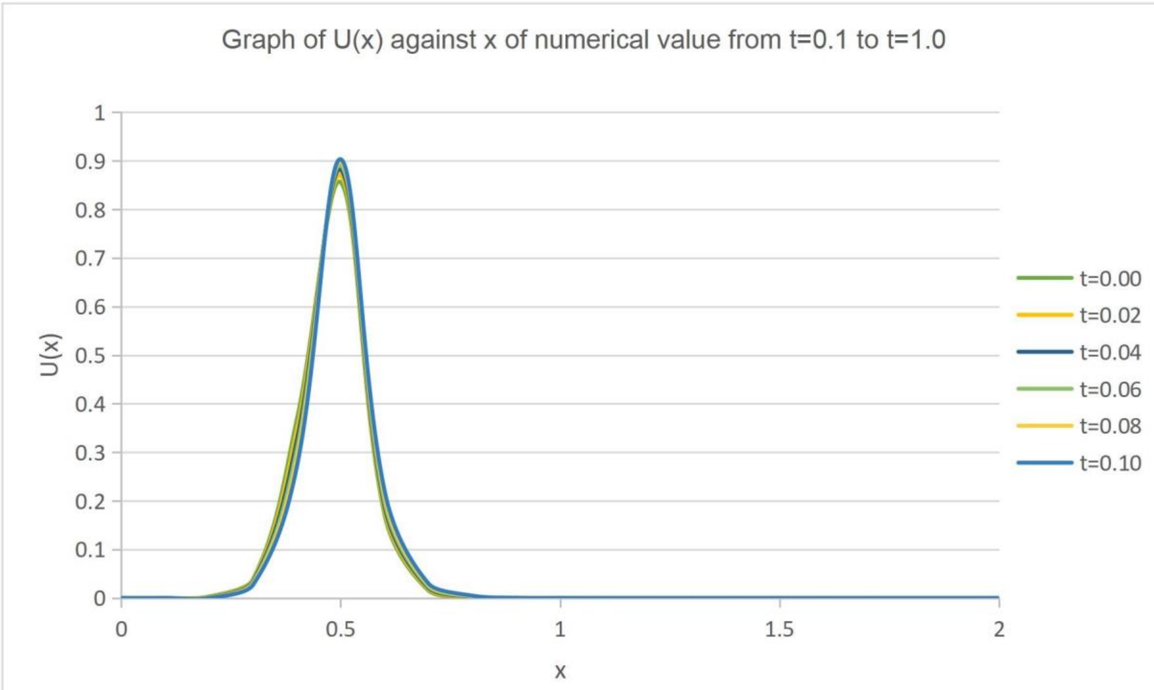


Figure 3: heat map representing the solution of the KdV equation

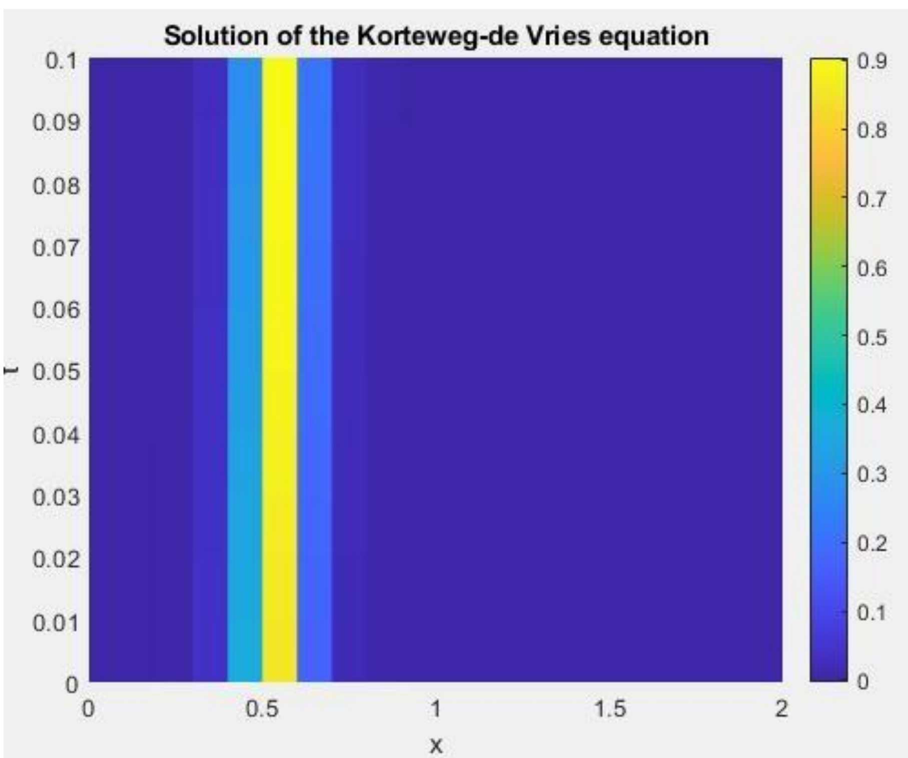


Figure 4: the 3D graph of the solution of the KdV equation

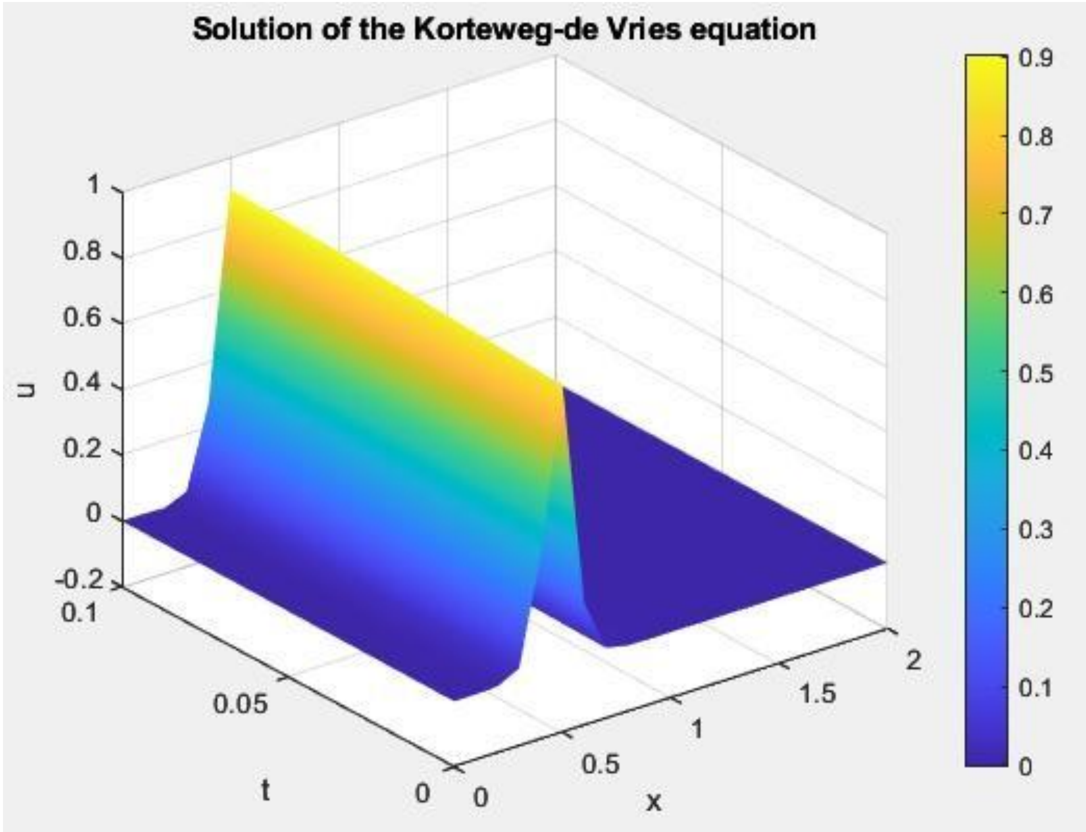
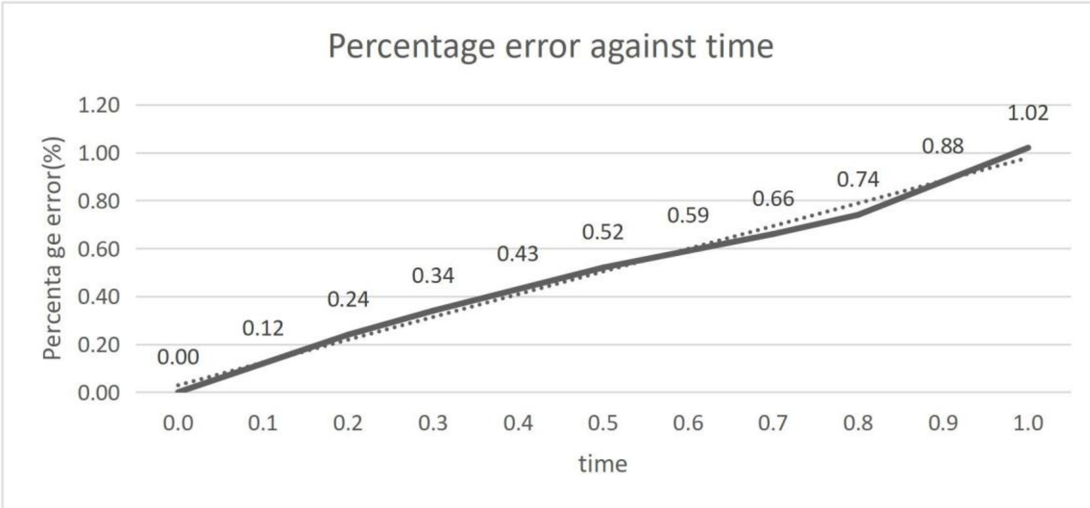


Figure 4.3 is a heat map representing the solution of the KdV equation and Figure 4.4 shows the 3D graph of the solution of the KdV equation. As shown in figure 4.3 where the value of *U* is at peak when its nearer to x=0.5 as supported by the 3D graph. This shown that the graph representing the KdV model properly.

Figure 5: the graph of percentage error against time



The percentage error increased as the time increased as shown in the Figure 5. Hence, big time step and value not recommended as it can commit a very big percentage error and caused the numerical to be inacccurate.the observation can be made that the error counted for each value of t is discrete enough to be acceptable. Thus, the objective of the study has been achieved as the RK4 method has been given an accurate value compared to the analytical solution.

Conclusion

The study of the KdV equation has led to a deeper understanding of the behavior of waves and nonlinear phenomena in general.It is a vital tool in numerous branches of physics and engineering since its soliton solutions have been seen in a variety of physical systems, such as water waves, optical fibres, and plasma waves. In this paper, The objectives were to achieve at a numerical solution that closely resembles the wave behaviour predicted by the KdV equation over a certain period of time, solve Korteweg-de Vries (KdV) equation by using MOL, to implement RK4 method to solve the system of ODEs in MOL and to determine the accuracy of the numerical solution of the KdV equation compared to the analytical solution.. The objectives of the study has been achieved through result and observation. The limitation in this study is that the researcher are facing with the difficulties in finding reading materials as reference regarding the KdV equation and how to implement it with MOL and its numerical solution. Researcher that are interested to conduct the same research in the future later on regarding the topic of KdV equation through MOL could use another numerical method such as Euler's method to solve the system of ODE as there's not much paper conducted about this topic. It would benefit future mathematician who want to explore this topic even more.

Acknowledgement
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