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Infestation of Golden Mussel Model by using Differential Equations

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Abstract
The growth of invasive species, Golden Mussel (*Limnorperna Fortunei*), poses a significant ecological threat to freshwater ecosystem worldwide. This project studies the mathematical modelling, specifically differential equations, to formulate a system capturing key parameter such as the population growth of mussel and larvae and their interactions with native species, algae. In addition, Routh-Hurwitz Criterion is applied to investigate the stability of equilibrium points of the model. Besides, numerical simulations are also conducted in this project to show the stability of the system. Subsequently, the basic reproduction number (R_0) is calculated to assess the potential for mussel population establishment. The calculation of R_0 provides a quantitative measure of the mussel's ability to establish a population in a given environment. Through this combined approach, this project aims to identify critical threshold and the conditions that may lead to the growth of Golden Mussel populations. The outcomes of this research contribute not only to the specific context of the Golden Mussel infestation but also demonstrate the applicability of differential equations, the Routh-Hurwitz Criterion, and R_0 in analyzing ecological system. This triple-methods approach provide a comprehensive framework for modelling, stability analysis, and assessing the potential establishment of the invasive species in freshwater environments.

Keywords: Mathematics Modelling, Differential Equations, Basic Reproduction Number, Larvae-Mussel-Algae Interaction

Introduction

This project will focus on the application of mathematics in ecosystem stability. The project is conducted to study the mathematical model of interaction of prey predator in ecosystem modelling. The species that been took as model in this project is the presence of the golden mussel (*Limpnoperna Fortunei*) in Brazil. The mathematical model produced can be used as the infestation control of the density of golden mussel that affect the imbalance of ecosystem. For this project, I had studied the paper of "Analysis of a Mathematical Model for Golden Mussel Infestation" [1].

Golden mussel is one of the members in Mytilidae family where it is a sessile bivalve mollusk species. The colours of its shell are with dark-brown and paler-yellow brown. Golden mussel lives in lakes and also rivers. The adult mussel can grow up until 3 to 4 cm in its size, with shells consisting of two valves. It has strong filtration rates and have the ability to fertilized and form colonies in a short period and achieve capacity with a density around 150,000 individuals/m^2 [2]. According to Nakano *et al* [3], golden mussel can survive up to 2 to 3 years. Golden mussel can filter the microalgae on the surface of water with amount that more than its own weight [4].

It is found that the golden mussel is a filter-feeding species of marine mollusk from southern China post 1960 [5]. Around 1990, it was introduced to Japan and South America and caused high frequency of research on it [6]. From there, it spread rapidly and cover most of the Río de la Plata basin, as well as the basins of Guaíba and Tramandaí (Brazil), Patos–Mirim (Brazil–Uruguay), and Mar Chiquita (central Argentina) (Oliveira *et al*, 2015). It dispersed to north through Paraná, Uruguay and Paraguay Rivers [7]. In the year of 2021, a report verified that the Upper Uruguay River is fully covered by the invader, golden mussel, involving standing water environment like lakes or ponds, flowing water environments such as rivers or streams [8].

Literature Review

Golden mussels have high ability in adaption to different environment, including low dissolved oxygen, high flow velocity and even heavy water pollution [9]. It can harm economic sector such as wastewater procession plant [10] and operation in aquatic generator such as wall corrosion and raising of flow resistance within tunnels [5]. To mitigate the impact of infestation, the Brazilian subsidiary of China Three Gorges Corporation (CTG), signed an agreement of initiated the "Golden Mussel Treatment Project" as part of research and development program of the Brazilian Electricity Regulatory Agency (ANEEL) [11].

In recent researches, many authors proposed the model based on the interaction between mussel and algae using differential equations, such as in paper written by Shen *et al* [12]. However, in this project, interaction intraspecies in between adult mussel and it juvenile will be addressed. Our aim is to study the existence of equilibrium points and their stability analysis from a model. This mathematical model is modified from a particular case of the diffusion-convection model presented in Silva et al. [13]

Mathematical Formulation

The definition of ODEs is an equation involving derivative of one or more dependent variables with respect to one or more independent variables [14]. The model of larvae-mussel-algae interaction is governed by the equations:

$$\frac{dL}{dT} = r_1 M\left(1 - \frac{L}{K_L}\right) - \left(b_1 + \gamma\right)L \, , \tag{1a}$$

$$\frac{dM}{dT} = \left(\frac{A^2}{c_1^2 + A^2}\right)\gamma L\left(1 - \frac{M}{K_M}\right) - b_2 M \, , \tag{1b}$$

$$\frac{dA}{dT} = r_2 A\left(1 - \frac{A}{K_A}\right) - b_3\left(\frac{A^2}{c_2^2 + A^2}\right)M \, . \tag{1c}$$

In the above model, *L* represents larvae population, *M* represents mussel population and *A* represents algae population. Equation (1a) describes the population of larvae is depended on its own birth rate from mussel, r_1, but limited by the its own carrying capacity, K_L. Besides, its loss if given by the sum of the larval mortality rate, b_1, and the maturation rate, by, γ, growing up become mussel. For function equation (1b), it denoted the increment of population of mussel is due to the maturation of larvae, γ, and the ability of algae as food supply. The mussel population is limited by its own carrying capacity, K_M , and the mortality rate of mussel, b_2. Likewise, the algae population is influence by its growth rate, r_2, and limited by its own carrying capacity, K_A. The loss of algae is modeled by the mortality rate by predation by mussel, b_3 .

Nondimensionalization

When the governing equations of a problem are known, the nondimensionalization of these equations (applied in their classical form) is a useful and widely used method that can be used to identify the dimensionless groups that rule the solution [15].

The process of nondimensionalization is used in this project to simplify the model to make the analysis easier. This process transformed the model in equation (1a), (1b) and (1c) with variables and parameters to a dimensionless form model. By letting $l = \frac{L}{K_L}, m = \frac{M}{K_M}, a = \frac{A}{K_A}$ and $t = T$. By using product rule, the dimension model is derived into dimensionless model as follow:

$$\frac{dl}{dt} = \frac{dl}{dL} \bullet \frac{dL}{dT} \bullet \frac{dT}{dt} \tag{2a}$$

$$\frac{dm}{dt} = \frac{dm}{dM} \bullet \frac{dM}{dT} \bullet \frac{dT}{dt} \tag{2b}$$

$$\frac{da}{dt} = \frac{da}{dA} \bullet \frac{dA}{dT} \bullet \frac{dT}{dt} \tag{2c}$$

Finally, the dimensionless equations can be reduced to:

$$\frac{dl}{dt} = \alpha_1 (1 - l) m - \beta_1 l \tag{3a}$$

$$\frac{dm}{dt} = \alpha_2 \left(\frac{a^2}{\xi_1 + a^2} \right) (1 - m) l - \beta_2 m \tag{3b}$$

$$\frac{da}{dt} = \alpha_3 a (1 - a) - \beta_3 \left(\frac{a^2}{\xi_2 + a^2} \right) m \tag{3c}$$

with all the parameters are positive

Table 1: Parameters

Dimensionless Parameter	***Dimensional Parameter***
α_1	$\frac{r_1 K_M}{K_L}$
β_1	$(b_1 + \gamma) K_L$
α_2	$\frac{\gamma K_L}{K_M}$
β_2	$b_2 K_M$
α_3	$\frac{r_2}{K_A}$
β_3	$b_3 K_m$
ξ_1	$\frac{c_1^2}{K_A^2}$
ξ_2	$\frac{c_2^2}{K_A^2}$

Equilibrium Points

An equilibrium point in the state space is a point at which the rates-of-change for all the state variable are zero. By considering equation (2a), (2b) and (2c), to calculate the equilibrium solution, the equations become as below:

$$\alpha_1 (1 - l) m - \beta_1 l = 0 \tag{4a}$$

$$\alpha_2 \left(\frac{a^2}{\xi_1 + a^2} \right) (1 - m) l - \beta_2 m = 0 \tag{4b}$$

$$\alpha_3 a (1 - a) - \beta_3 \left(\frac{a^2}{\xi_2 + a^2} \right) m = 0 \tag{4c}$$

Solve the equations simultaneously. As result, the equilibrium points are given as: E_1 $(0,0,0)$, E_2 $(0,0,1)$ and $E_3 (l^*, m^*, a^*)$ with

$$l^* = \frac{\alpha_1\alpha_2 a^{*2} + \xi_1\beta_1\beta_2 + \beta_1\beta_2 a^{*2}}{\alpha_1\alpha_2 a^{*2} - \alpha_2\beta_1 a^{*2}} \tag{5}$$

$$m^* = \frac{\alpha_1\alpha_2 a^{*2} + \xi_1\beta_1\beta_2 + \beta_1\beta_2 a^{*2}}{\alpha_1\alpha_2 a^{*2} + \xi_1\alpha_1\beta_2 + \alpha_1\beta_2 a^{*2}} \tag{4}$$

and a^* satisfies

$$\phi_1 a^{*5} + \phi_2 a^{*4} + \phi_3 a^{*3} + \phi_4 a^{*2} + \phi_5 a^* + \phi_6 = 0 \tag{5}$$

where

$$\begin{aligned}
\phi_1 &= -\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_3\beta_2 \\
\phi_2 &= \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\beta_2 \\
\phi_3 &= -\alpha_1\alpha_2\alpha_3\xi_2 - \alpha_1\alpha_3\xi_2\beta_2 - \alpha_1\alpha_3\xi_1\beta_2 - \alpha_1\alpha_2\beta_3 - \beta_1\beta_2\beta_3 \\
\phi_4 &= \alpha_1\alpha_2\alpha_3\xi_2 + \alpha_1\alpha_3\xi_2\beta_2 + \alpha_1\alpha_3\xi_1\beta_2 \\
\phi_5 &= -\alpha_1\alpha_3\xi_1\xi_2\beta_2 - \xi_1\beta_1\beta_2\beta_3 \\
\phi_6 &= \alpha_1\alpha_3\xi_1\xi_2\beta_2
\end{aligned}$$

The first equilibrium point obtained is the trivial equilibrium points, $E_1 = (0,0,0)$. This equilibrium point represents the extinction of all populations in the ecosystem. Next, E_2 $(0,0,1)$ contributed to the existence of algae and the extinction of larvae and mussel. In addition, this equilibrium point also denotes as the non-infestation scenario by golden mussel and only contained algae. Lastly, equilibrium point of infestation state, $E_3 = (l^*, m^*, a^*)$ shows the scenario of existence of all population and mussel and larvae exist in the ecosystem by getting nutrient from algae.

Basic Reproduction Number

There are some research papers using basic reproduction rate as an indicator to discuss the stability pattern of the predator-prey model in paper [16] and [17]. Determinants of the basic reproductive number of the prey can be seen as a threshold between the possibility of infestation or extinction.
Since mussel and larvae are in one type of species act as invader to this system, thus derive the equation (5) in term of *m* and substitute and combine into equation (6).

$$\left[\alpha_2\left(\frac{a^2}{\xi_1 + a^2}\right)(1-m)\left(\frac{\alpha_1}{\alpha_1 m - \beta_1}\right) - \beta_2\right]m = 0 \tag{6}$$

To obtain the basic reproduction number at predator-free equilibrium point $(l, m, a) = (0,0,1)$, given that:

$$F = \alpha_2\left(\frac{1}{\xi_1 + 1}\right)\left(\frac{\alpha_1}{-\beta_1}\right) \tag{7}$$

and

$$V = -\beta_2 \tag{8}$$

with using the formula of $R_0 = FV^{-1}$. The basic reproduction number in dimensionless form is denoting as follow:

$$R_0 = \frac{\alpha_1\alpha_2}{(\xi_1 + 1)\beta_1\beta_2} \tag{9}$$

The number shows the potential of spread of mussel within a given population. If R_0 is greater than 1, it leads to exponential growth in the number where the golden mussel will infest in the time period, while in the other hand, the golden mussel will reduce to extinct state when R_0 is less than 1. In additional, R_0 equal to 1 express the stability of all the population where the invader will exist but maintain in the same density of population.

Methodology

Stability Analysis

In this project report, two types of methods are used to analysis the stability of the system, Routh-Hurwitz Criterion and numerical simulations for some chosen parameters of the equilibrium points.

Routh-Hurwitz Criteria

To study the stability of equilibrium points, Routh-Hurwitz Criteria with $n = 3$ is used. The Jacobian of the model is given by:

$$J = \begin{bmatrix} -\alpha_1 m - \beta_1 & \alpha_1(1-l) & 0 \\ \alpha_2\left(\dfrac{a^2}{\xi_1 + a^2}\right)(1-m) & -\alpha_2\left(\dfrac{a^2}{\xi_1 + a^2}\right)l - \beta_2 & \dfrac{2\alpha_2(1-m)la}{\xi_1 + a^2} - \dfrac{2\alpha_2(1-m)la^3}{\left(\xi_1 + a^2\right)^2} \\ 0 & -\beta_3\left(\dfrac{a^2}{\xi_2 + a^2}\right) & \alpha_3(1-a) - \alpha_3 a - \dfrac{2\beta_3 am}{\xi_2 + a^2} + \dfrac{2\beta_3 a^3 m}{(\xi_2 + a^2)^2} \end{bmatrix} \tag{10}$$

The first equilibrium point obtained is $E_1(0,0,0)$, which is trivial equilibrium point. The Jacobian at $E_1(0,0,0)$:

$$J(0,0,0) = \begin{bmatrix} -\beta_1 & \alpha_1 & 0 \\ 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \tag{11}$$

with the characteristics equation $C(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, where

$$a_1 = -\alpha_3 + \beta_1 + \beta_2, \qquad a_2 = (-\beta_1 - \beta_2)\alpha_3 + \beta_1\beta_2, \qquad a_3 = -\alpha_3\beta_1\beta_2,$$

Since α_3 is always negative for every parameter values, thus this equilibrium point is always unstable.

For second equilibrium point, $E_2(0,0,1)$, the Jacobian is as follow:

$$J(0,0,1) = \begin{bmatrix} -\beta_1 & 1 & 0 \\ \dfrac{\alpha_1}{2} & -\beta_2 & 0 \\ 0 & -\dfrac{1}{2} & -1 \end{bmatrix} \tag{12}$$

with the characteristics equation $P(\lambda) = \lambda^3 + d_1\lambda^2 + d_2\lambda + d_3$, where

$$d_1 = \alpha_3 + \beta_2 + \beta_1 \qquad d_2 = \frac{\alpha_3(2\beta_1 + 2)(\xi_1 + 1) + \beta_1\beta_2(\xi_1 + 1) - \alpha_1\alpha_2}{(\xi_1 + 1)} \qquad d_3 = \frac{\left(\beta_1\beta_2(\xi_1 + 1) - \alpha_1\alpha_2\right)\alpha_3}{(\xi_1 + 1)}$$

It is noticed that $d_1 > 0$ since all the parameters are positive. Next $d_3 > 0$ and $d_1 d_2 - d_3 > 0$ when $\dfrac{\alpha_1\alpha_2}{\beta_1\beta_2(\xi_1 + 1)} < 1$. Thus, we can conclude that the equilibrium point, $E_2(0,0,1)$ is asymptotically stable when $R_0 < 1$.

Numerical Simulations

The numerical simulations in this section served as a study case for the infestation of the golden mussel and demonstrate the stability of the equilibrium points discussed. Parameters chosen were obtained from equations (5), (6), (7) and the basic reproduction number, R_0 in equation (1). It is found that some of the parameters have symmetrical relationship with their respective pair where likewise showed in equation (11). For example, parameter α_1 and β_2 are in opposite proportional relationship towards R_0. If β_2 is kept constant, an increment of α_1 leads to corresponding increment in R_0. Meanwhile, R_0 will act in opposite way with β_2 when α_1 is kept constant. However, this response is only function without involving other parameters. By observing that the parameter β_2 directly influenced R_0, it will serve as foundation in this numerical simulation to invest the relationship between the stability and R_0.

Table 2: Parameter values used in Figure 1 and Figure 2

Parameter	Simulation 1	Simulation 2
α_1	2.15	2.15
β_1	1.42	1.42
α_2	3.22	3.22
β_2	4.9	0.1
α_3	3.75	3.75
β_3	54.76	54.76
ξ_1	0.01	0.01
ξ_2	0.81	0.81
R_0	0.9851	0.9851

The increasing and decreasing of β_2 will directly influence the stability of the equilibrium points and also indicates the potential of the infestation of golden mussel by effecting the result of R_0. Thus, the parameter values used in simulation 2 are the same as simulation 1 except β_2 which as shown in Table 2.

Results and discussion

From the results using Routh-Hurwitz Criterion and the numerical simulations, the model showed different stability depend on the development of mussel. As considered before, the stability can be determined by the basic reproduction number and influenced by the maturation rate of mussel which consist in parameter β_2.

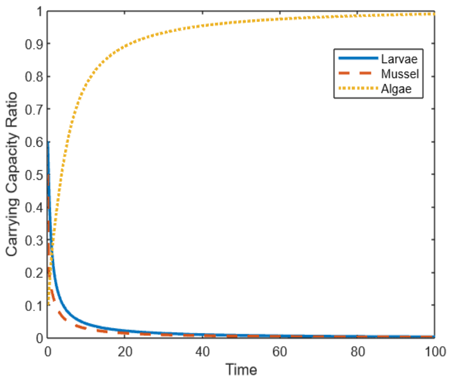


Figure 1: Three Population vs Time at $\beta_2 = 4.9$ and $R_0 = 0.9851$

Figure 1 shows the behaviour of the system with basic reproduction rate (R_0) equal to 0.9851. The initial values used for larvae, mussel and algae populations were set as 0.6,0.5 and 0.1 respectively. Figure 1 shows the extinction of the larvae and mussel, while algae population tend to grow up until reached it carrying capacity limit. This revealed a situation where mussel was under a struggle conditions. As result, it was observed a decay of population for invaders, larvae and mussel, they tend towards situation of extinction. In this other hand, population of algae was developed to its maximum capacity without the infestation of mussel. Figure 1 shows $E_2(0,0,1)$ is stable when $R_0 < 1$.

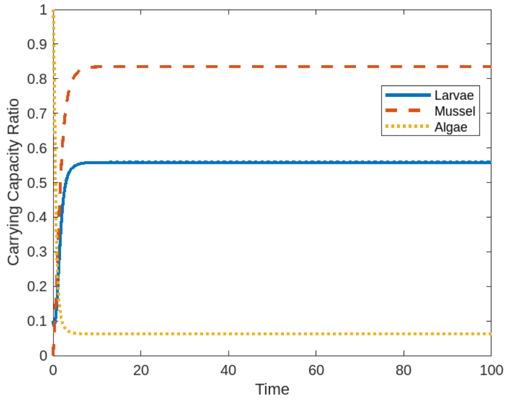


Figure 2 Three Population vs Time at $\beta_2 = 0.1$ and $R_0 = 48.2708$

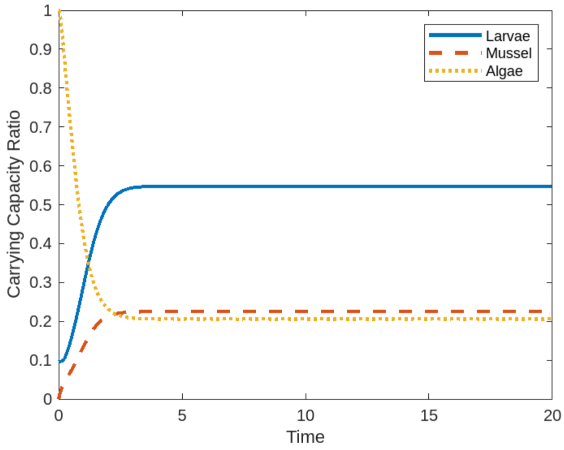
Figure 2 shows the population of larvae, mussel and algae over time when the basic reproduction rate (R_0) equal to 48.2708. This is an important consideration, as shown in Table 2, the reducing of β_2 to 0.1 was directly influenced the result of R_0 and the final populations level. In this case, the initial conditions for larvae, mussel and algae populations were set as 0.1, 0.001 and 1 respectively. According to the finding above, this numerical simulation shows the infestation state of mussel. Figure 2 shows us the populations of larvae and mussel increase and reach their equilibrium values at a certain time, while population of algae which act as their primary food sources declined until it stabilized to $a*$.

As a result, it can conclude that $E_2 = (0,0,1)$ is stable when $R_0 < 1$ and coexistence point, $E_3 = (l*, m*, a*)$ is stable when $R_0 > 1$. To strengthen the evidence, different parameters in basic reproduction number $\left(R_0 = \frac{\alpha_1 \alpha_2}{(\xi_1 + 1)\beta_1 \beta_2} \right)$ are used to determine the stability of the system. Table 3 represent the parameter values used in the next simulations.

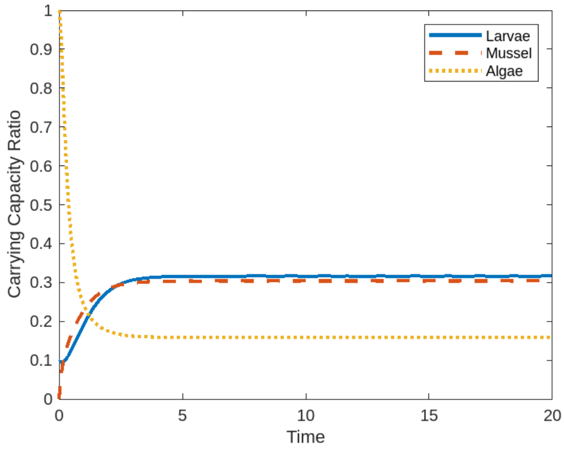
Table 3: Parameter Values included in R_0

Parameter		α_1	β_1	α_2	β_2	α_3	β_3	ξ_1	ξ_2	R_0
Control		2.15	1.42	3.22	4.9	3.75	54.76	0.01	0.81	0.9851
Changing Variables	α_1	**7.6**	1.42	3.22	4.9	3.75	54.76	0.01	0.81	3.4823
	α_2	2.15	1.42	**9.5**	4.9	3.75	54.76	0.01	0.81	2.9064
	β_1	2.15	**0.3**	3.22	4.9	3.75	54.76	0.01	0.81	4.6629

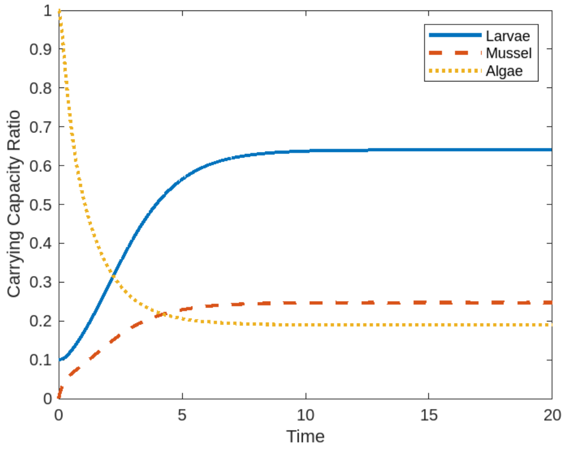
Figures 3(a), 3(b), and 3(c) present the trend of population variation of larvae, mussel and algae with different values of parameters α_1, α_2 and β_1 as provided in Table 3. Those parameters influence the values of basic reproduction number. Parameter values in Table 2 is used as benchmark against other test results are measured with $R_0 = 0.9851$. The control test shows the second equilibrium point, $E_2 = (0,0,1)$ is stable when $R_0 < 1$. As result, it is observed that Figure 3 illustrates the stability of coexistence points, $E_3 = (l*, m*, a*)$ when $R_0 > 1$.



a) $\alpha_1 = 7.6$ and $R_0 = 48.2708$



b) $\alpha_2 = 9.5$ and $R_0 = 2.9064$



c) $\beta_1 = 0.3$ and $R_0 = 4.6629$

Figure 3 Three Populations vs Time for Table 3

Figure 3(a) shows the population by increasing the values of α_1 which is the larvae growth rate $\left(\alpha_1 = \frac{r_1 K_M}{K_L}\right)$ with $R_0 = 3.4823$. It is found that the equilibrium point of larvae population is exceedingly higher when compare with population of mussel and algae.

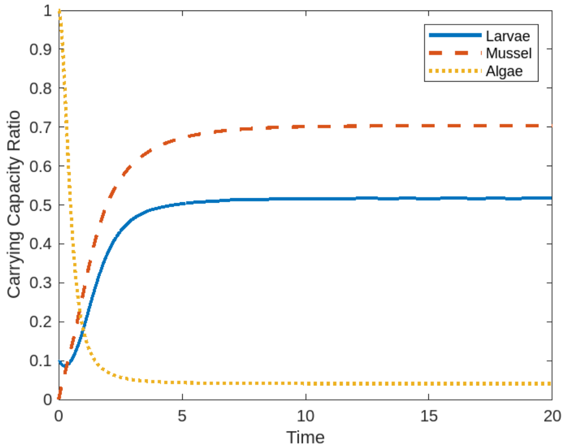
Figure 3(b) shows three populations trend with increase the values of α_2 which is the larvae maturation rate $\left(\alpha_2 = \frac{\gamma K_L}{K_M}\right)$ with $R_0 = 2.9064$. The increasing of maturation rate of larvae reduced the larvae population since it is mature to mussel in a higher rate compare to Figure 3(a). At the same time, the equilibrium point of mussel population in Figure 3(b) is higher than Figure 3(a).

Figure 3(c) shows three populations trend with increase the values of β_1 which is the larvae maturation rate and larvae mortality rate $\left(\beta_1 = (b_1 + \gamma)K_L\right)$ with $R_0 = 4.6629$. When the larvae maturation rate is fixed, the decreasing of mortality rate of larvae increase the larvae population in its stability compared to Figure 3(b).

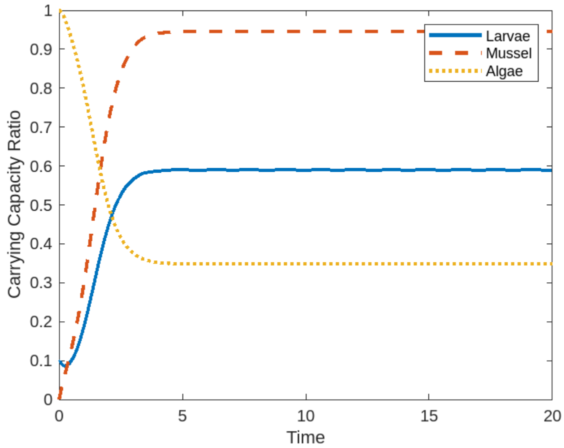
Table 4: Parameter Values excluded from R_0

Parameter		α_1	β_1	α_2	β_2	α_3	β_3	ξ_1	ξ_2	R_0
Control		2.15	1.42	3.22	4.9	3.75	54.76	0.01	0.81	48.2708
Changing Variables	α_3	2.15	1.42	3.22	4.9	**2.02**	54.76	0.01	0.81	48.2708
	β_3	2.15	1.42	3.22	4.9	3.75	**6.9**	0.01	0.81	48.2708

Next, by checking the effect of parameters excluded in basic reproduction number, α_3 and β_3, the values are shown in Table 4. Opposed to Table 3, the control test used in following is the parameter values in Figure 2. It showed the system is stable at coexistence point, $E_3 = (l^*, m^*, a^*)$ at $R_0 > 1$.



a) $\alpha_3 = 2.02$ and $R_0 = 48.2708$



b) $\beta_3 = 6.9$ and $R_0 = 48.2708$

Figure 4 Three Population vs Time for Table 4

It is observed that there is no effect on the stability of system when modifying the parameter values of α_3 and β_3. With the modification, the state of system is remained the same with the control system where both are stable at coexistence point $E_3 = (l^*, m^*, a^*)$.

Conclusion

The model highlights the influence of algae in the development of the mussel population. First of all, three equilibrium points of the model were found on based on the dimensionless system, which represent different biological outcomes. Extinction of all population, $E_1(0,0,0)$, only algae survive, $E_2(0,0,1)$ and coexistence of all population, $E_3(l^*, m^*, a^*)$. For the first situation, there is no independent population involve in the ecosystem which represent trivial point. The result of stability analysis of this point is locally unstable. For second equilibrium point, $E_2(0,0,1)$,the only survivor population is the algae. From the stability analysis, it is found that this point is asymptotically stable when basic reproductive number (R_0) is smaller than 1. From numerical simulation, the algae is prevalence when $R_0 < 1$. The stability of coexistence point for all population is also conducted using numerical simulations with condition $R_0 > 1$. By changing one of the parameters in R_0, it is found that the coexistence point is stable when $R_0 > 1$.

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