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One-Dimensional Pollutant Transport in Groundwater

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Abstract
Groundwater pollution can happen when human activities, factories, graveyards, mining waste, and other factors contaminate underground water. In order to clean up the contaminated area, it is very important to understand how the pollutants move in the groundwater. Based on the mathematical model, this study focuses on finding an analytical solution to a one-dimensional advection-diffusion equation (ADE) representing the pollutant transport to understand how pollutants spread in two different distinct geological formations namely sand and clay. The pollutant concentration is considered in liquid as well as in solid phases using the Laplace transform method. The graphical interpretation analyzes the behavior of the concentration distribution on solute transport such as the contaminant concentration distribution with different geological formations, diffusion parameter values, velocity parameter values and time values. It is found that the pollutant moves faster in sand instead of clay. Also, the pollutant moves faster when the seepage velocity and dispersion coefficient are increased while the pollutant moves slower as time increases. The findings of this research can be invaluable as an initial predictive tool for the planning of groundwater resource management and remediation projects.
Keywords: Groundwater, Geological Formation, ADE, Laplace Transform

Introduction
Groundwater is a crucial source of water supply, defined as water beneath the land surface, typically found in aquifers composed of sedimentary rocks like sands, clays, gravels, sandstones, and limestones. Groundwater quality is influenced by geological formations, structures, chemical weathering, and contamination. Contaminants in groundwater vary widely, including physical, inorganic, organic chemicals, and radioactive substances [1]. Ensuring a safe water supply requires effective remediation strategies and mathematical models play a crucial role in understanding pollutant movement, predicting migration patterns, and assessing risks. These models, such as the Advection-Difussion Equation (ADE), are essential for developing strategic plans to mitigate groundwater contamination challenges.

Diffusion is a molecule that travels from a high concentration location to a low concentration area [2] while advection refers to solute displacement along the main flow direction, while dispersion involves solute spreading in longitudinal and transversal directions due to complex pore structures. The ADE, based on mass conservation and Fick's law of diffusion states that diffusion rate is proportional to the concentration gradient and is essential for modeling hydrodynamic dispersion in porous media [3]. Hydrodynamic dispersion is the term used to describe the dispersing of solutes or pollutants in the fluid as a result [4]. Mechanical dispersion and diffusion combine in groundwater flow to create hydrodynamic dispersion. Mechanical dispersion reflects the fact that not everything in the porous medium travels at the average water flow speed. Some paths are faster, some slower, some longer, some shorter.

Previously, other research papers have been conducted to solve ADE, such as analytical solutions for the (ADE) with variable dispersion coefficient and velocity are derived using the Green's function method (GFM) [7] while [8] concentrated to a one-dimensional model for pollutant transport in both homogeneous and heterogeneous semi-infinite groundwater reservoirs of these coefficients reflects the heterogeneity of hydrogeological media while considering Dirichlet-type and Nuemann-type boundary conditions. Other than that, a numerical approach for modeling the advection-diffusion equation, employing a method that combines Laplace transform (LT) and Chebyshev spectral collocation method (CSCM) [9] while a numerical algorithm based are developed on the Laplace transform and the numerical inverse Laplace transform for numerical modeling of diffusion problems[10]. Next, the solutions for the (ADE) with temporal coefficients are derived for a pollutant's point source moving linearly along the axis of a one-dimensional semi-infinite domain [11] and an analytical solution is achieved for the two-dimensional ADE with variable coefficients in a semi-infinite, heterogeneous porous medium [12].

This study aims to understand pollutant transport in sand and clay geological formations. The objectives are to develop a one-dimensional ADE model for groundwater contamination considering zero production and first-order decay, derive an analytical solution using the Laplace transform method, and analyze concentration behavior for varying diffusion and velocity parameters. Basically, this study is conducted based on the work by Singh et al.[4].

Mathematical Model and Its Analytical Solution

From Singh *et al.* [4], the governing equation consists of ADE and the source term is in the form:

$$\frac{\partial c}{\partial t}+\frac{\rho(1-n)}{n}\frac{\partial F}{\partial t}=D\frac{\partial^2 c}{\partial x^2}-u\frac{\partial c}{\partial x}-\mu c+\gamma\,, \tag{1}$$

where D is a longitudinal dispersion coefficient, c is the volume-averaged dispersing solute concentration in the liquid phase, F is the volume-averaged dispersing solute concentration in the solid phase, u is the unsteady uniform downward pore seepage velocity, x is a longitudinal direction of flow, t is a time, γ is the zero production rate coefficient for solute production in the liquid phase, μ is the first-order decay rate coefficient in the liquid phase, ρ is the bulk density of the porous medium and n is the porosity of geological formation.

Contaminant goes from solid phase into liquid phase under the linear isotherm condition.

$$F=K_d c, \tag{2}$$

where K_d is the distribution coefficient. Distribution coefficient can be defined as the concentration adsorbed by the solid phase to the liquid phase into groundwater reservoir.

In this research, the growth of solute along the space initially was a linear combination of the initial concentration taken into consideration. The initial condition can be written as:

$$c(x,0)=c_i+\frac{\gamma x}{u} \qquad x>0, t=0. \tag{3}$$

Mixed type of boundary condition in the splitting time domain at the source due to increasing human activity at the earth's surface and the solute concentration in groundwater increases in time. Along the boundary conditions as below can be written as

$$-D\frac{\partial c}{\partial x}+uc=uc_0 \qquad x=0,\ 0<t<t_0\,,$$

$$-D\frac{\partial c}{\partial x}+uc=0 \qquad x=0, t>0 \tag{4}$$

Due to no mass flow at the other end of the domain, a flux type boundary condition can be written as

$$\frac{\partial c}{\partial x}=0, \qquad x=L\,, t>0. \tag{5}$$

Using Equation (2), Equation (1) can be written as:

$$R\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{dx^2} - u\frac{\partial c}{\partial x} - \mu c + \gamma, \tag{6}$$

where

$$R = 1 + \frac{1-n}{n}\rho K_d \tag{7}$$

From the dispersion theory, dispersion is directly proportional to seepage velocity which can be written as $D \propto u$. Hence, $D = Au$, where, A is a constant that depends upon the pore geometry of the groundwater. By letting a temporally dependent coefficient, the coefficients can be written as

$$u = u_0\psi(mt),$$

$$D = D_0\psi(mt),$$

$$\mu = \mu_0\psi(mt),$$

$$\gamma = \gamma_0\psi(mt),$$

where u_0 and D_0 is the initial seepage velocity and initial dispersion coefficient respectively while μ_0 and γ_0 is the initial first-order decay rate coefficient and initial zero-order production rate coefficient for solute production in the liquid phase respectively and $\psi(mt)$ is a non-dimensional expression where is the flow resistance coefficient, the Equation (6) can be expressed as

$$\frac{R}{\psi(mt)}\frac{\partial c}{\partial t} = D_0\frac{\partial^2 c}{\partial x^2} - u_0\frac{\partial c}{\partial x} - \mu_0 c + \gamma_0. \tag{8}$$

A new time variable T^*

$$T^* = \int \psi(mt)\ dt \tag{9}$$

is introduced and Equation (8) can be expressed as

$$R\ \frac{\partial c}{\partial T^*} = D_0\frac{\partial^2 c}{\partial x^2} - u_0\frac{\partial c}{\partial x} - \mu_0 c + \gamma_0\ . \tag{10}$$

The following non-dimensional variables are introduced

$$C = \frac{c}{c_0}; X = \frac{nx}{L}\ ; T = \frac{T^* n u_0}{L}; T_p = \frac{n u_0 t_0}{L}\ ;\ \mu^* = \frac{\mu_0 L}{n u_0};\ \gamma^* = \frac{\gamma_0 L}{n c_0 u_0} \tag{11}$$

which reduce Equation (10) into

$$R\frac{\partial C}{\partial t} = \frac{1}{Pe}\frac{\partial^2 C}{\partial X^2} - \frac{\partial C}{\partial X} - \mu^* C + \gamma^*, \tag{12}$$

where

$$Pe = \frac{L u_0}{n D_0}. \tag{13}$$

While the initial and boundary conditions (3) to (5) becomes

$$C(X,T) = \frac{c_i}{c_0} + \gamma^* X\ . \qquad X > 0,\ T = 0 \tag{14}$$

$$-\frac{1}{Pe}\frac{\partial C}{\partial X} + C = 1 \qquad X = 0,\ 0 \le T \le T_p$$

$$-\frac{1}{Pe}\frac{\partial C}{\partial X} + C = 0 \qquad X = 0,\ T > T_p \tag{15}$$

$$\frac{\partial C}{\partial X} = 0 \qquad X = n,\ T > 0. \tag{16}$$

The transformation is used to reduce equation (12) to become

$$C(X,T) = K(X,T)\exp\left[\frac{X}{2}Pe - \frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)T\right] + \frac{\gamma^*}{\mu^*} \tag{17}$$

$$R\frac{\partial K}{\partial T} = \frac{1}{Pe}\frac{\partial^2 K}{\partial X^2} \tag{18}$$

Meanwhile, from transformation (17), the initial condition in (14) and boundary condition in (15) and (16) become

$$C(X,0) = \left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*} + \gamma^* X\right)e^{-\frac{X}{2}Pe} \quad X > 0, T = 0 \tag{19}$$

$$-\frac{1}{Pe}\frac{\partial K}{\partial X} + \frac{K}{2} = \left(1 - \frac{\gamma^*}{\mu^*}\right)e^{\left[\frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)T\right]} \quad X = 0, 0 < T \le T_p \tag{20}$$

$$-\frac{1}{Pe}\frac{\partial K}{\partial X} + \frac{K}{2} = \left(-\frac{\gamma^*}{\mu^*}\right)e^{\left[\frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)T\right]} \quad X = 0, T > T_p \tag{21}$$

$$\frac{\partial K}{\partial X} = -\frac{K}{2}Pe \quad X = n, T \ge 0 \tag{22}$$

Taking Laplace integral transform technique

$$\bar{K}(X,P) = \beta_1 e^{\sqrt{PeRPX}} + \beta_2 e^{\sqrt{PeRPX}} - \left(\frac{\gamma^*}{\mu^*} - \frac{c_i}{c_0}\right)\frac{1}{\left(P - \frac{Pe}{4R}\right)}\ e^{-\frac{Pe}{2}X} + \ \gamma^*\left(X - \frac{1}{R\left(P - \frac{Pe}{R}\right)}\right)e^{-\frac{Pe}{2}X}, \tag{23}$$

where,

$\bar{K}(X,P) = \int_0^\infty K(X,T)e^{-PT}\ dT, \beta_1\ and\ \beta_2$ are the arbitrary constants.

$\beta_2 = A + B + D + E$

$\beta_1 = \sqrt{\frac{R}{Pe}}\frac{\gamma^*}{R}\frac{1}{\left(P - \frac{Pe}{4R}\right)(\sqrt{P} + \xi)}e^{-\frac{Pe}{2}n}e^{-\sqrt{PeRPn}} + \beta_2\left[1 - \frac{2\xi}{(\sqrt{P} + \xi)}\right]e^{-2\sqrt{PeRPn}}$

where,

$\xi = \frac{1}{2}\sqrt{\frac{Pe}{R}}\ and\ Q = -\frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)$

The analytical solution based on the paper by Singh et al. [4] can be followed as

$$\begin{aligned} C(X,T) = [&F(X,T) + G(X,T) + H(X,T) + I(X,T) + J(X,T) + M(X,T) + N(X,T) + P(X,T) + S(X,T) \\ &+ U(X,T)]\exp\left[\frac{X}{2}Pe - \frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)T\right] + \frac{\gamma^*}{\mu^*}, \\ &0 < T \le T_p \end{aligned} \tag{24}$$

$$\begin{aligned} C(X,T) = \big[&F(X,T) + G(X,T) + \{H(X,T) - H(X,T - T_p)\} + I(X,T) + J(X,T) + M(X,T) \\ &+ \{N(X,T) - N(X,T - T_p)\} + P(X,T) + S(X,T) + U(X,T)\big]\exp\left[\frac{X}{2}Pe - \frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)T\right] \\ &+ \frac{\gamma^*}{\mu^*}, \quad T > T_p \end{aligned} \tag{25}$$

where

$$F(X,T) = -\frac{\gamma^*}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)}C_1 - \frac{1}{2(a-\xi)}D_1\right\} - \frac{\gamma^*}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\ \frac{\xi}{(a^2-\xi^2)}\exp\{\xi^2T+\xi Z\}\, erfc\left\{\frac{Z}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$C_1 = \exp(a^2T - aZ)\, erfc\left(\frac{Z}{2\sqrt{T}} - a\sqrt{T}\right)$$

$$D_1 = \exp(a^2T + aZ)\, erfc\left(\frac{Z}{2\sqrt{T}} + a\sqrt{T}\right)$$

$$Z = n\sqrt{RPe} - X\sqrt{RPe}$$

$$a = \frac{1}{2}\sqrt{\frac{Pe}{R}}$$

$$G(X,T) = G_1(X,T) + G_2(X,T) - G_3(X,T) + G_4(X,T)$$

$$G_1(X,T) = -\frac{\gamma^*}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)}C_2 - \frac{1}{2(a-\xi)}D_2\right\} - \frac{\gamma^*}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\ \frac{\xi}{(a^2-\xi^2)}\exp\{\xi^2T+\xi Y\}\, erfc\left\{\frac{Y}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$G_2(X,T) = \frac{4\gamma^*\xi}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)^2}C_2 + \frac{1}{2(a-\xi)^2}D_2\right\} + \frac{4\gamma^*\xi}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\ \frac{2\xi T}{(a^2-\xi^2)}A\ - \frac{4\gamma^*\xi}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{a^2+\xi^2}{(a^2-\xi^2)^2} + \frac{2\xi^2T+\xi Y}{(a^2-\xi^2)}\right\}\ \exp\{\xi^2T+\xi Y\}\ \ erfc\left\{\frac{Y}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$C_2 = \exp(a^2T - aY)\, erfc\left(\frac{Y}{2\sqrt{T}} - a\sqrt{T}\right)$$

$$D_2 = \exp(a^2T + aY)\, erfc\left(\frac{Y}{2\sqrt{T}} + a\sqrt{T}\right)$$

$$A = \frac{1}{\sqrt{\pi T}}\exp\left(-\frac{Y^2}{4T}\right)$$

$$Y = 3n\sqrt{RPe} - X\sqrt{RPe}$$

$$G_3(X,T) = \frac{\gamma^*}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)}C_3 - \frac{1}{2(a-\xi)}D_3\right\} + \frac{\gamma^*}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\ \frac{\xi}{(a^2-\xi^2)}\exp\{\xi^2T+\xi\omega\}\, erfc\left\{\frac{\omega}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$G_4(X,T) = \frac{8\gamma^*\xi}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)^2}C_3 + \frac{1}{2(a-\xi)^2}D_3\right\} + \frac{8\gamma^*\xi}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\ \frac{2\xi T}{(a^2-\xi^2)}A_1\ - \frac{8\gamma^*\xi}{\sqrt{PeR}}\ e^{-\frac{Pe}{2}n}\left\{\frac{a^2+\xi^2}{(a^2-\xi^2)^2} + \frac{2\xi^2T+\xi\omega}{(a^2-\xi^2)}\right\}\ \exp\{\xi^2T+\xi\omega\}\ \ erfc\left\{\frac{\omega}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$C_3 = \exp(a^2T - a\omega)\, erfc\left(\frac{\omega}{2\sqrt{T}} - a\sqrt{T}\right)$$

$$D_3 = \exp(a^2T + a\omega)\, erfc\left(\frac{\omega}{2\sqrt{T}} + a\sqrt{T}\right)$$

$$A_1 = \frac{1}{\sqrt{\pi T}}\exp\left(-\frac{\omega^2}{4T}\right)$$

$$\omega = 5n\sqrt{RPe} - X\sqrt{RPe}$$

$$H(X,T) = H_1(X,T) - H_2(X,T) + H_3(X,T)\ - H_4(X,T)$$

$$H_1(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)(C_4 + D_4)$$

$$H_2(X,T) = \sqrt{\frac{Pe}{R}}\left(1-\frac{\gamma^*}{\mu^*}\right)2\xi\left\{\frac{1}{2(\sqrt{Q}+\xi)}C_4 - \frac{1}{2(\sqrt{Q}-\xi)}D_4\right\} + 2\xi\sqrt{\frac{Pe}{R}}\left(1-\frac{\gamma^*}{\mu^*}\right)\frac{\xi}{Q-\xi^2}\exp\{\xi^2T+\xi Y_1\}\ erfc\left\{\frac{Y_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$C_4 = \exp\left(QT - \sqrt{Q}Y_1\right) erfc\left(\frac{Y_1}{2\sqrt{T}} - \sqrt{QT}\right)$$

$$D_4 = \exp\left(QT + \sqrt{Q}Y_1\right) erfc\left(\frac{Y_1}{2\sqrt{T}} + \sqrt{QT}\right)$$

$$Y_1 = 2n\sqrt{RPe} - X\sqrt{RPe}$$

$$H_3(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)(C_5 + D_5)$$

$$H_4(X,T) = 6\xi\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)\left\{\frac{1}{2(\sqrt{Q}+\xi)}C_5 - \frac{1}{2(\sqrt{Q}-\xi)}D_5\right\} + 6\xi\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)\frac{\xi}{Q-\xi^2}\exp\{\xi^2 T + \xi\omega_1\}\ erfc\left\{\frac{\omega_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$C_5 = \exp\left(QT - \sqrt{Q}\omega_1\right) erfc\left(\frac{\omega_1}{2\sqrt{T}} - \sqrt{QT}\right)$$

$$D_5 = \exp\left(QT + \sqrt{Q}\omega_1\right) erfc\left(\frac{\omega_1}{2\sqrt{T}} + \sqrt{QT}\right)$$

$$\omega_1 = 4n\sqrt{RPe} - X\sqrt{RPe}$$

$$Q = \frac{1}{R}\left(\frac{Pe}{4} + \mu^*\right)$$

$$I(X,T) = I_1(X,T) - I_2(X,T) + I_3(X,T) - I_4(X,T)$$

$$I_1(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)(C_6 + D_6)$$

$$I_2(X,T) = 2\xi\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)\left\{\frac{1}{2(a+\xi)}C_6 - \frac{1}{2(a-\xi)}D_6\right\} + 2\xi\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)\frac{\xi}{a^2-\xi^2}\exp\{\xi^2 T + \xi Y_1\}\ erfc\left\{\frac{Y_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$C_6 = \exp(a^2 T - aY_1)\, erfc\left(\frac{Y_1}{2\sqrt{T}} - a\sqrt{T}\right)$$

$$D_6 = \exp(a^2 T + aY_1)\, erfc\left(\frac{Y_1}{2\sqrt{T}} + a\sqrt{T}\right)$$

$$Y_1 = 2n\sqrt{RPe} - X\sqrt{RPe}$$

$$I_3(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)(C_7 + D_7)$$

$$I_4(X,T) = 6\xi\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)\left\{\frac{1}{2(a+\xi)}C_7 - \frac{1}{2(a-\xi)}D_7\right\} + 6\xi\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)\frac{\xi}{a^2-\xi^2}\exp\{\xi^2 T + \xi\omega_1\}\ erfc\left\{\frac{\omega_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$C_7 = \exp(a^2 T - a\omega_1)\, erfc\left(\frac{\omega_1}{2\sqrt{T}} - a\sqrt{T}\right)$$

$$D_7 = \exp(a^2 T + a\omega_1)\, erfc\left(\frac{\omega_1}{2\sqrt{T}} + a\sqrt{T}\right)$$

$$\omega_1 = 4n\sqrt{RPe} - X\sqrt{RPe}$$

$$J(X,T) = J_1(X,T) - J_2(X,T) - J_3(X,T) - J_4(X,T) + J_5(X,T) - J_6(X,T) - J_7(X,T) + J_8(X,T)$$

$$J_1(X,T) = -\frac{1}{2}\frac{\gamma^*}{\sqrt{RPe}}(C_6 + D_6)$$

$$J_2(X,T) = -\gamma^*\frac{2\xi}{\sqrt{RPe}}\left\{\frac{1}{2(a+\xi)}C_6 - \frac{1}{2(a-\xi)}D_6\right\} - \gamma^*\frac{2\xi}{\sqrt{RPe}}\frac{\xi}{a^2-\xi^2}\exp\{\xi^2 T + \xi Y_1\}\ erfc\left\{\frac{Y_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$J_3(X,T) = -\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}(2aT-Y_1)C_6\right\} - \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}(2aT+Y_1)D_6\right\}$$

$$J_4(X,T) = \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,2\xi\,\frac{1-(a+\xi)(2aT-Y_1)}{4a(a+\xi)^2}C_6 + \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,2\xi\,\frac{(2aT+Y_1)(a-\xi)-1}{4a(a-\xi)^2}D_6 - \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,2\xi\,\frac{T}{a^2-\xi^2}A_2 \quad + \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}2\xi\,\frac{\xi}{(a^2-\xi^2)^2}\,\exp(\xi^2T+\xi Y_1)\,erfc\left\{\frac{Y_1}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$J_5(X,T) = -\frac{1}{2}\frac{\gamma^*}{\sqrt{RPe}}(C_7+D_7)$$

$$J_6(X,T) = -\gamma^*\frac{6\xi}{\sqrt{RPe}}\left\{\frac{1}{2(a+\xi)}C_7 - \frac{1}{2(a-\xi)}D_7\right\} - \gamma^*\frac{6\xi}{\sqrt{RPe}}\frac{\xi}{a^2-\xi^2}\exp\{\xi^2T+\xi\omega_1\}\;erfc\left\{\frac{\omega_1}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$J_7(X,T) = -\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}(2aT-\omega_1)C_7\right\} - \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}(2aT+\omega_1)D_7\right\}$$

$$J_8(X,T) = 6\xi\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,\frac{1-(a+\xi)(2aT-\omega_1)}{4a(a+\xi)^2}C_7 + 6\xi\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,2\xi\,\frac{(2aT+\omega_1)(a-\xi)-1}{4a(a-\xi)^2}D_7 - 6\xi\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,\frac{T}{a^2-\xi^2}A_3 \quad + 6\xi\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\frac{\xi}{(a^2-\xi^2)^2}\,\exp(\xi^2T+\xi\omega_1)\,erfc\left\{\frac{\omega_1}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$A_2 = \frac{1}{\sqrt{\pi T}}\exp\left(-\frac{Y_1^2}{4T}\right)$$

$$A_3 = \frac{1}{\sqrt{\pi T}}\exp\left(-\frac{\omega_1^2}{4T}\right)$$

$$M(X,T) = M_1(X,T) + M_2(X,T) - M_3(X,T) + M_4(X,T)$$

$$M_1(X,T) = -\frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)}C_8 - \frac{1}{2(a-\xi)}D_8\right\} - \frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\,\frac{\xi}{(a^2-\xi^2)}\,exp\,\{\xi^2T+\xi\sigma\}erfc\left\{\frac{\sigma}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$M_2(X,T) = \frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)^2}C_8 + \frac{1}{2(a-\xi)^2}D_8\right\} + \frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\,\frac{2\xi T}{(a^2-\xi^2)}A_4 - \frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\left\{\frac{a^2+\xi^2}{(a^2-\xi^2)^2} + \frac{2\xi^2T+\xi\sigma}{(a^2-\xi^2)}\right\}\exp\{\xi^2T+\xi\sigma\}\;erfc\left\{\frac{\sigma}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$M_3(X,T) = \frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)}C_9 - \frac{1}{2(a-\xi)}D_9\right\} + \frac{\gamma^*}{\sqrt{RPe}}\;e^{-\frac{Pe}{2}n}\,\frac{\xi}{(a^2-\xi^2)}\,exp\,\{\xi^2T+\xi\psi\}erfc\left\{\frac{\psi}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$M_4(X,T) = \frac{\gamma^*}{\sqrt{RPe}}6\xi e^{-\frac{Pe}{2}n}\left\{\frac{1}{2(a+\xi)^2}C_9 + \frac{1}{2(a-\xi)^2}D_9\right\} + \frac{\gamma^*}{\sqrt{RPe}}6\xi\;e^{-\frac{Pe}{2}n}\,\frac{2\xi T}{(a^2-\xi^2)}A_5 - \frac{\gamma^*}{\sqrt{RPe}}6\xi\;e^{-\frac{Pe}{2}n}\left\{\frac{a^2+\xi^2}{(a^2-\xi^2)^2} + \frac{2\xi^2T+\xi\psi}{(a^2-\xi^2)}\right\}\exp\{\xi^2T+\xi\psi\}\;\;erfc\left\{\frac{\psi}{2\sqrt{T}}+\xi\sqrt{T}\right\}$$

$$C_8 = \exp(a^2T-a\sigma)\,erfc\left(\frac{\sigma}{2\sqrt{T}}-a\sqrt{T}\right)$$

$$D_8 = \exp(a^2T+a\sigma)\,erfc\left(\frac{\sigma}{2\sqrt{T}}+a\sqrt{T}\right)$$

$$C_9 = \exp(a^2T-a\psi)\,erfc\left(\frac{\psi}{2\sqrt{T}}-a\sqrt{T}\right)$$

$$D_9 = \exp(a^2T+a\psi)\,erfc\left(\frac{\psi}{2\sqrt{T}}+a\sqrt{T}\right)$$

$$\sigma = n\sqrt{RPe} + X\sqrt{RPe}$$

$$\psi = 3n\sqrt{RPe} - X\sqrt{RPe}$$

$$A_4 = \frac{1}{\sqrt{\pi T}}\exp\left(-\frac{\sigma^2}{4T}\right)$$

$A_5 = \frac{1}{\sqrt{\pi T}} \exp\left(-\frac{\sigma^2}{4T}\right)$

$N(X,T) = N_1(X,T) + N_2(X,T) - N_3(X,T)$

$N_1(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)(C_{10} + D_{10})$

$C_{10} = \exp\left(QT - X\sqrt{RPeQ}\right) erfc\left(\frac{X}{2}\sqrt{\frac{RPe}{T}} - \sqrt{QT}\right)$

$D_{10} = \exp\left(QT + X\sqrt{RPeQ}\right) erfc\left(\frac{X}{2}\sqrt{\frac{RPe}{T}} + \sqrt{QT}\right)$

$N_2(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)(C_{11} + D_{11})$

$N_3(X,T) = 4\xi\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)\left\{\frac{1}{2(\sqrt{Q}+\xi)}C_{11} - \frac{1}{2(\sqrt{Q}-\xi)}D_{11}\right\} + 4\xi\sqrt{\frac{Pe}{R}}\left(1 - \frac{\gamma^*}{\mu^*}\right)\frac{\xi}{Q-\xi^2}\exp\{\xi^2 T + \xi\sigma_1\}\ erfc\left\{\frac{\sigma_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$

$C_{11} = \exp\left(QT - \sigma_1\sqrt{Q}\right) erfc\left(\frac{\sigma_1}{2\sqrt{T}} - \sqrt{QT}\right)$

$D_{11} = \exp\left(QT + \sigma_1\sqrt{Q}\right) erfc\left(\frac{\sigma_1}{2\sqrt{T}} + \sqrt{QT}\right)$

$\sigma_1 = 2n\sqrt{RPe} + X\sqrt{RPe}$

$P(X,T) = P_1(X,T) + P_2(X,T) - P_3(X,T)$

$P_1(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)(C_{12} + D_{12})$

$C_{12} = \exp\left(a^2 T - aX\sqrt{RPe}\right) erfc\left(\frac{X}{2}\sqrt{\frac{RPe}{T}} - a\sqrt{T}\right)$

$D_{12} = \exp\left(a^2 T + aX\sqrt{RPe}\right) erfc\left(\frac{X}{2}\sqrt{\frac{RPe}{T}} + a\sqrt{T}\right)$

$P_2(X,T) = \frac{1}{2}\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)(C_{13} + D_{13})$

$P_3(X,T) = 4\xi\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)\left\{\frac{1}{2(a+\xi)}C_{13} - \frac{1}{2(a-\xi)}D_{13}\right\} + 4\xi\sqrt{\frac{Pe}{R}}\left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*}\right)\frac{\xi}{a^2-\xi^2}\exp\{\xi^2 T + \xi\sigma_1\}\ erfc\left\{\frac{\sigma_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$

$C_{11} = \exp(a^2 T - a\sigma_1)\ erfc\left(\frac{\sigma_1}{2\sqrt{T}} - a\sqrt{T}\right)$

$D_{11} = \exp(a^2 T + a\sigma_1)\ erfc\left(\frac{\sigma_1}{2\sqrt{T}} + a\sqrt{T}\right)$

$S(X,T) = S_1(X,T) + S_2(X,T) - S_3(X,T) - S_4(X,T) + S_5(X,T) + S_6(X,T)$

$S_1(X,T) = -\frac{1}{2}\frac{\gamma^*}{\sqrt{RPe}}(C_{12} + D_{12})$

$S_2(X,T) = -\frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}\left(2aT - X\sqrt{RPe}\right)C_{12}\right\} - \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}\left(2aT + X\sqrt{RPe}\right)D_{12}\right\}$

$$S_3(X,T) = \frac{1}{2}\frac{\gamma^*}{\sqrt{RPe}}(C_{13} + D_{13})$$

$$S_4(X,T) = \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}(2aT - \sigma_1)C_{13}\right\} + \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\left\{\frac{1}{4a}(2aT + \sigma_1)D_{13}\right\}$$

$$S_5(X,T) = \frac{\gamma^*}{\sqrt{RPe}}4\xi\left\{\frac{1}{2(a+\xi)}C_{13} - \frac{1}{2(a-\xi)}D_{13}\right\} + \frac{\gamma^*}{\sqrt{RPe}}4\xi\frac{\xi}{a^2-\xi^2}\exp\{\xi^2T + \xi\sigma_1\}\, erfc\left\{\frac{\sigma_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$S_6(X,T) = \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,4\xi\,\frac{1-(a+\xi)(2aT-\sigma_1)}{4a(a+\xi)^2}C_{13} + \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,4\xi\,\frac{(2aT+\sigma_1)(a-\xi)-1}{4a(a-\xi)^2}D_{13} - \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}\,4\xi\frac{T}{a^2-\xi^2}A_6 + \frac{\gamma^*}{R}\sqrt{\frac{Pe}{R}}4\xi\frac{\xi}{(a^2-\xi^2)^2}\exp\{\xi^2T + \xi\sigma_1\}\, erfc\left\{\frac{\sigma_1}{2\sqrt{T}} + \xi\sqrt{T}\right\}$$

$$U(X,T) = \left(\frac{c_i}{c_0} - \frac{\gamma^*}{\mu^*} + \gamma^*X\right)e^{-\left(\frac{Pe}{2}X - \frac{Pe}{4R}T\right)} - \frac{\gamma^*T}{R}e^{-\left(\frac{Pe}{2}X - \frac{Pe}{4R}T\right)}$$

$$H(X,T) - H(X,T-T_p) = \{H_1(X,T) - H_1(X,T-T_p)\} - \{H_2(X,T) - H_2(X,T-T_p)\} + \{H_3(X,T) - H_3(X,T-T_p)\} - \{H_4(X,T) - H_4(X,T-T_p)\}$$

$$N(X,T) - N(X,T-T_p) = \{N_1(X,T) - N_1(X,T-T_p)\} + \{N_2(X,T) - N_2(X,T-T_p)\} - \{N_3(X,T) - N_3(X,T-T_p)\}$$

Result and Discussion

The following data in Table 1 is from [6] is used in MATLAB coding to obtain the concentration contamination from equation (3.17) by assuming $K(X,T) = 1$ for simplification.

Table 1: The Parameter and Its Value

Parameter	Value
c_i	0.01
c_0	1.0
u_0	$0.01\ (myear^{-1})$
D_0	$0.1(m^2year^{-1})$
γ_0	0.000001
μ_0	$0.0005(year^{-1})$
L	$200m$
K_d	0.0025
Pe	2.0
ρ	999
m	$0.004(year^{-1})$
k	0.2
$n\ sand$	0.37
$n\ clay$	0.55

The average porosity of different geological formations is n=0.37 for sand and n=0.55 for clay. From Figure 1, it is evident that contaminant concentration increases with distance in both sand and clay, but the rate of increase differs. In sand, the concentration rises more rapidly, indicating faster dispersion compared to clay, possibly due to higher permeability or a more efficient transport mechanism.

Figure 2 shows contaminant concentrations in sand and clay at T=1 and T=10. In sand, the concentration increases rapidly with distance at both times, while in clay, it increases more slowly, showing a more linear growth. At any given distance and time, contaminant concentration is higher in sand than in clay, indicating that sand allows faster dispersion. Over time, both materials show an increase in concentration, but it is more pronounced in sand.

Figure 1 Concentration Behaviour of Sand vs Clay

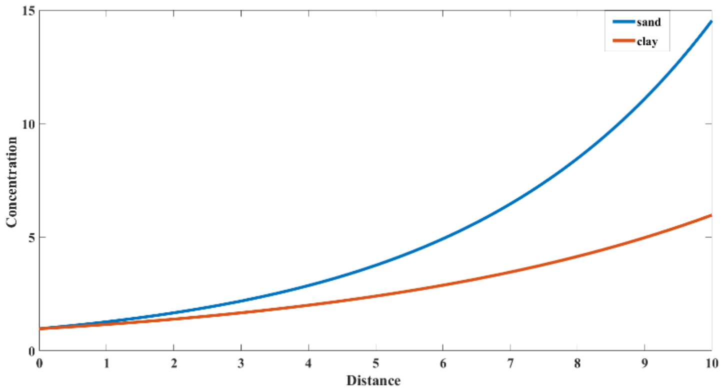
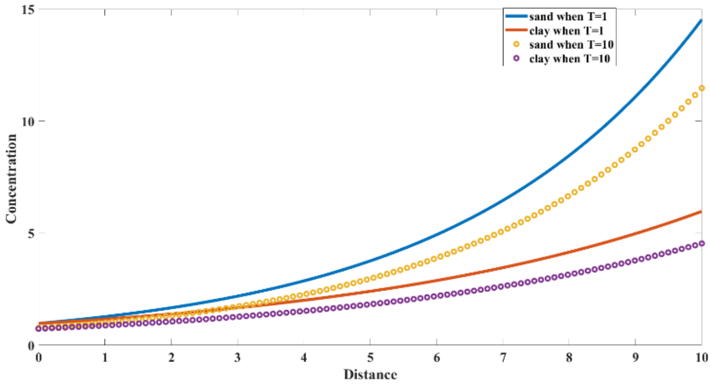


Figure 2 Concentration Behaviour of Sand vs Clay at Different Time



Also,two graphs have been observed in the sand geological formation, showing variations in seepage velocity and dispersion coefficient. From Figure 3, contaminant concentration is shown with seepage velocities of u=0.01, u=0.015, and u=0.02. At u=0.01, there is a gradual increase in concentration with distance. At u=0.015, the increase is steeper, and at u=0.02, the increase is the steepest, indicating that higher seepage velocities result in higher contaminant concentrations. From Figure 4, contaminant concentration is shown with dispersion coefficients of D=0.1, D=0.15, and D=0.2. At D=0.2, there is the steepest increase in concentration with distance. At D=0.15, the increase is steeper compared to D=0.1, indicating that higher dispersion coefficients result in faster dispersion. In both cases, concentration increases more rapidly with distance.

Figure 3 Concentration Behaviour of Sand with Different Seepage Velocity

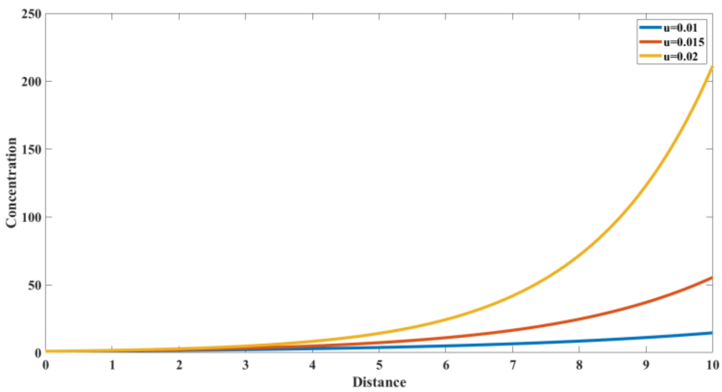
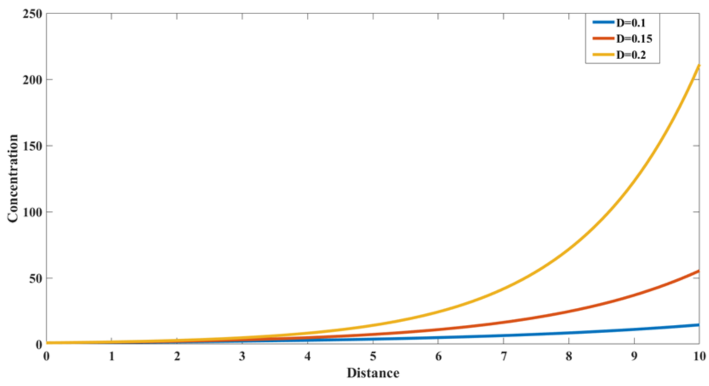


Figure 4 Concentration Behaviour of Sand with Different Dispersion Coefficient



Two graphs have been observed in the clay geological formation, showing variations in seepage velocity and dispersion coefficient. From Figure 5, contaminant concentration is shown with seepage velocities of u=0.01, u=0.015, and u=0.02. At u=0.01, there is a gradual increase in concentration with distance. At u=0.015, the increase is steeper, and at u=0.02, it is the steepest, indicating higher seepage velocities result in higher contaminant concentrations. From Figure 6, contaminant concentration is shown with dispersion coefficients of D=0.1, D=0.15, and D=0.2. At D=0.2, there is the steepest increase in concentration with distance. At D=0.15, the increase is steeper compared to D=0.1, indicating higher dispersion coefficients result in faster dispersion. In both cases, concentration increases more rapidly with distance.

Figure 5 Concentration Behaviour of Clay with Different Seepage Velocity

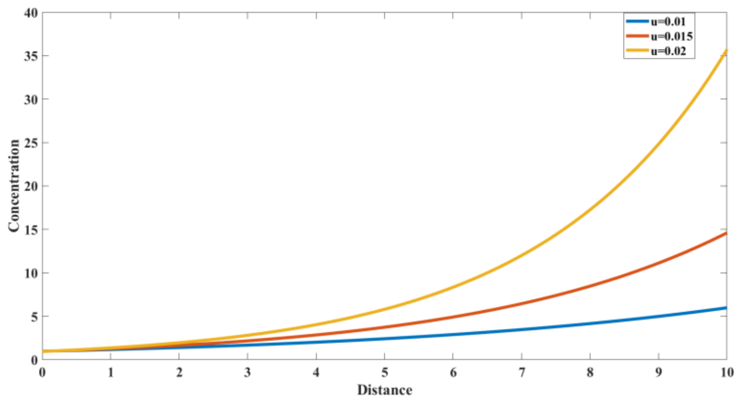
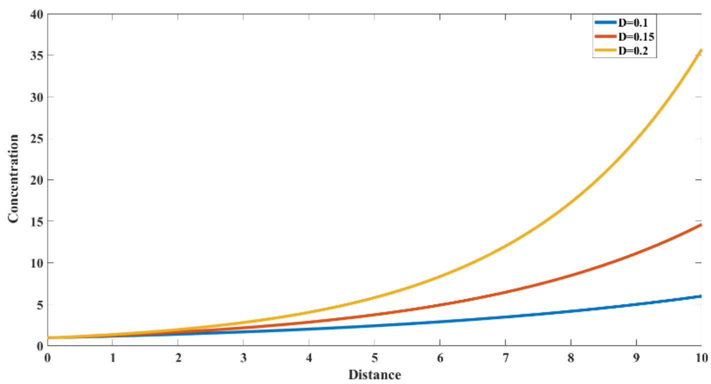


Figure 6 Concentration Behaviour of Clay with Different Dispersion Coefficient



Conclusion

This study highlights the importance of using the ADE model to solve groundwater contamination problems and protect water resources, which are crucial for all living things. The study analytically solves the ADE for pollutant transport using the Laplace transform technique and analyzes contaminant behavior in different geological formations like sand and clay. Sand shows faster contaminant movement compared to clay, with dispersion coefficients and seepage velocities significantly influencing the rate and distribution of contamination. Higher values result in more rapid contaminant spread. The study's findings, achieved through analytical solutions and graphical interpretations using MATLAB, emphasize the need for continuous monitoring and understanding of contaminant dynamics to protect water resources and guide future research and management strategies.

Solving the ADE for different formations, dispersion coefficients, and velocities helps examine concentration distribution behaviour, aiding in cleanup strategy decisions. Integrating these models into decision-making enhances the protection of water resources, public health, and sustainable water supplies. The research findings serve as a valuable predictive tool for groundwater resource management and remediation projects, emphasizing the need for ongoing research, interdisciplinary collaboration, and adaptive management for successful outcomes.

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