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The General Zeroth-Order Randić Index of Non-Braid Graph of a Commutative Ring

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Abstract
The non-braid graph of a ring R, denoted by Γ_R, is a simple graph with a vertex set $R \backslash B(R)$, where $B(R)$ is the set of x where $x \in R$ such that $xyx = yxy$ for all $y \in R$. Two distinct vertices x and y are adjacent if and only if $xyx \neq yxy$. This paper focuses on finding the general formula for the general zeroth-order Randić index of the non-braid graph of $\mathbb{Z}_n$. The study involves reviewing previous literature to infer new results. The outcomes include measurements of the degree of each vertex and the general zeroth-order Randić index for different values of n.
Keywords: General zeroth-order Randić index, Non-braid graph, Commutative ring.

Introduction

Initially, researchers focused on topological indices for graphs with chemical structures, but they later expanded their studies to include graphs in general. Furthermore, the idea of deriving graphs from algebraic structures is gaining acceptance. Consequently, topological indices of graphs can be computed using algebraic structures like groups, rings and module. Several types of graphs that can be used as representations of a commutative ring are the zero-divisor graph, the prime graph, the prime ideal graph, and the non-braid graph [1-4]. The concept of non-braid graph was introduced by Cahyati et al. [4] in 2022. Let R be a ring and $B(R) = \{x \in R \mid \forall y \in R, xyx = yxy\}$. A non-braid graph of ring R, denoted by Γ_R, is a simple graph whose vertex set is $R \backslash B(R)$. Two distinct vertices x and y are connected by an edge if and only if $xyx \neq yxy$. Furthermore, x and y that satisfy this condition are denoted by $x \sim y$. One type of topological indices is the general zeroth-order Randić index, which was developed by Li and Zheng [5]. The general zeroth-order Randić index is defined as the sum of the degree of each vertex raised to the power of $\alpha \neq 0$. In 2024, Ismail et al. constructed general formulas for the general zeroth-order Randić index of a zero-divisor graph using the set of integers modulo p^n [6].

This research presents general formulas for the general zeroth-order Randić index of a non-braid graph using the set of integers modulo p and $2p$. The general zeroth-order Randić index was determined for the cases $\alpha = 1, 2$, and 3.

Preliminaries

Before heading to the main results of this research, some necessary definitions, theorems and lemma for the discussion are provided.

Definition 2.1 [4] Let R be a ring and $B(R) = \{x \in R \mid \forall y \in R, xyx = yxy\}$. A non-braid graph of ring R, denoted by Γ_R, is a simple graph whose vertex set is $R \backslash B(R)$. Two distinct vertices x and y are connected by an edge if and only if $xyx \neq yxy$. Furthermore, x and y that satisfy this condition are denoted by $x \sim y$.

Definition 2.2 [7] The degree of $v \in V(G)$ is the number of neighbours of v, denoted by $\deg(v)$.

Definition 2.3 [7] The number of vertices of graph G is its order, written as $|G|$; its number of edges denoted by $|E|$.

Theorem 2.4 [4] If p is prime, then $\Gamma_{\mathbb{Z}_n}$ is a complete graph.

Lemma 2.5 [4] If $n \in \mathbb{N}$, then

$$B(\mathbb{Z}_n) = \begin{cases} \{0, m\}, n = 2m \textit{ where } m \textit{ is odd}, \\ \{0\}, \quad \textit{otherwise}. \end{cases}$$

Theorem 2.6 [4] If $n \geq 3$, then $\Gamma_{\mathbb{Z}_n}$ is a connected graph.

Definition 2.7 [5] Let Γ be a connected graph and $\deg(u)$ be the degree of vertex u in the graph. Then, the general zeroth-order Randić index is

$$R_\alpha^0 = \sum_{u \in V} (\deg(u))^\alpha.$$

Results and discussion

This section presents some results related to the general zeroth-order Randić index of non-braid graph of integers modulo ring.

The first case explain about ring of integers modulo p.

Based on Theorem 2.4 and Lemma 2.5, $\Gamma_{\mathbb{Z}_n}$ is a complete graph with $|\mathbb{Z}_n \backslash B(\mathbb{Z}_n)| = n - 1$.

Corollary 3.1 Let n is prime, $x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)$. Then $\deg(x) = n - 2$.

The next proposition states the number of edges in the graph.

Proposition 3.2 If n be a prime, then the number of edges of $\Gamma_{\mathbb{Z}_n}$ is $\frac{(n-1)(n-2)}{2}$.

Proof. In Handshaking lemma, any graph has half as many edges as the sum of the degrees of all its vertices are proven. Using Corollary 3.1, we have

$$\left|E\left(\Gamma_{\mathbb{Z}_n}\right)\right| = \frac{1}{2} \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} \deg(x) = \frac{1}{2}(n-1)(n-2) = \frac{(n-1)(n-2)}{2}.$$

■

Next, the main results in this research, the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ for $\alpha = 1{,}2$ and 3 are presented in the following theorems.

Theorem 3.3 If n be a prime, then the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ when $\alpha = 1$ is $n^2 - 3n + 2$.

Proof. By using Definition 2.7 and Corollary 3.1

$$R_1^0 = \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} \deg(x) = (n-1)(n-2) = n^2 - 3n + 2.$$

■

Theorem 3.4 If n be a prime, then the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ when $\alpha = 2$ is $n^3 - 5n^2 + 8n - 4$.

Proof. By using Definition 2.7 and Corollary 3.1

$$R_2^0 = \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} (\deg(x))^2 = (n-1)(n-2)^2 = n^3 - 5n^2 + 8n - 4.$$

■

Theorem 3.5 If n be a prime, then the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ when $\alpha = 3$ is $n^4 - 7n^3 + 18n^2 - 20n + 8$.

Proof. By using Definition 2.7 and Corollary 3.1

$$R_3^0 = \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} (\deg(x))^3 = (n-1)(n-2)^3 = n^4 - 7n^3 + 18n^2 - 20n + 8.$$

■

Example 3.1 Let $\mathbb{Z}_5 = \{0,1,2,3,4\}$ then $\mathbb{Z}_5 \backslash B(\mathbb{Z}_5) = \{1,2,3,4\}$. So, a non-braid graph $\Gamma_{\mathbb{Z}_5}$ illustrated by Figure 3.1.

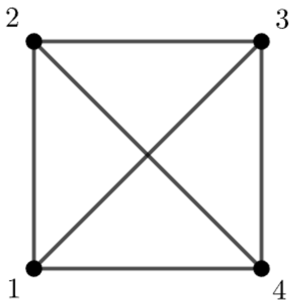

Figure 3.1 $\Gamma_{\mathbb{Z}_5}$

From Figure 3.1, we get $\deg(1) = 3 = \deg(2) = \deg(3) = \deg(4)$. Then,

$$\left|E\left(\Gamma_{\mathbb{Z}_5}\right)\right| = \frac{1}{2}\sum_{x \in \mathbb{Z}_5 \backslash B(\mathbb{Z}_5)} \deg(x) = \frac{\deg(1) + \deg(2) + \deg(3) + \deg(4)}{2}$$

$$= \frac{3 + 3 + 3 + 3}{2} = 6.$$

The general zeroth-order Randic index of $\Gamma_{\mathbb{Z}_5}$ for $\alpha = 3$ by manual calculations as follows.

$$R_3^0 = \sum_{x \in \mathbb{Z}_5 \backslash B(\mathbb{Z}_5)} (\deg(x))^3 = (\deg(1))^3 + (\deg(2))^3 + (\deg(3))^3 + (\deg(4))^3$$

$$= 3^3 + 3^3 + 3^3 + 3^3 = 108.$$

Now, we move to the second case for ring of integers modulo $2p$.

This proposition states the degree of vertex in the graph.

Proposition 3.6 Let $n = 2p$ with p be a prime and $x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)$. Then $\deg(x) = n - 4$.

Proof. Let $\mathbb{Z}_n$ be the integer modulo ring with $n = 2p$, where p is prime number. Based on Lemma 2.5 $|\mathbb{Z}_n \backslash B(\mathbb{Z}_n)| = n - 2$. Next, divide the vertices into two sets $V_1 = \{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n) \mid x = 2l, l \in \mathbb{N}\}$ with $|V_1| = p - 1$ and $V_2 = \{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n) \mid x \neq 2l, l \in \mathbb{N}\}$ with $|V_2| = p - 1$.

Because $\mathbb{Z}_n$ commutative ring, rewrite $xyx \neq yxy$ as $xy(x - y) \neq 0 \; mod \; n$. Now, divide into several cases as shown below.

Case 1 Take any $x, y \in V_1$. Clearly $xy(x - y) \neq 0 \; mod \; n$, then $x \backsim y$.

Case 2 Take any $x, y \in V_2$. Clearly $xy(x - y) \neq 0 \; mod \; n$, then $x \backsim y$.

Case 3 Take any $x \in V_1$ where $x = 2l$ and choose $y \in V_2$ where $y = x + p$.

$xy(x - y) = (2l)(2l + p)(-p) = (2p)m = 0 \; mod \; n$, $m \in \mathbb{N}$. Then $x \nsim y$.

Case 4 Due to Case 3, therefore take any $x \in V_2$ is not adjacent to exactly one $y \in V_1$.

Based on the above case, every $x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)$ is not adjacent to exactly one $y \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)$ and certainly not adjacent to x itself. Then $\deg(x) = (n - 2) - 2 = n - 4$ for every $x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)$.

■

In the following proposition, the number of edges in the graph is determined.

Proposition 3.7 If $n = 2p$ with p be a prime, then the number of edges of $\Gamma_{\mathbb{Z}_n}$ is $\frac{(n-2)(n-4)}{2}$.

Proof. Using the Handshaking lemma, and Proposition 3.6, we have

$$\left|E\left(\Gamma_{\mathbb{Z}_n}\right)\right| = \frac{1}{2}\sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} \deg(x) = \frac{1}{2}(n - 2)(n - 4) = \frac{(n - 2)(n - 4)}{2}$$

■

The last three theorems, explain about the general zeroth-order Randić index for cases $\alpha =$ 1,2 and 3.

Theorem 3.8 If $n = 2p$ with p be a prime, then the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ when $\alpha = 1$ is $n^2 - 6n + 8$.

Proof. By using Definition 2.7 and Proposition 3.6

$$R_1^0 = \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} \deg(x) = (n - 2)(n - 4) = n^2 - 6n + 8.$$

■

Theorem 3.9 If $n = 2p$ with p be a prime, then the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ when $\alpha = 2$ is $n^3 - 10n^2 + 32n - 32$.

Proof. By using Definition 2.7 and Proposition 3.6

$$R_2^0 = \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} (\deg(x))^2 = (n-2)(n-4)^2 = n^3 - 10n^2 + 32n - 32.$$

■

Theorem 3.10 If $n = 2p$ with p be a prime, then the general zeroth-order Randić index of $\Gamma_{\mathbb{Z}_n}$ when $\alpha = 3$ is $n^4 - 14n^3 + 72n^2 - 160n + 128$.

Proof. By using Definition 2.7 and Proposition 3.6

$$R_3^0 = \sum_{x \in \mathbb{Z}_n \backslash B(\mathbb{Z}_n)} (\deg(x))^3 = (n-2)(n-4)^3 = n^4 - 14n^3 + 72n^2 - 160n + 128.$$

■

Example 3.2 Let $\mathbb{Z}_{10} = \{0,1,2,3,4,5,6,7,8,9\}$, then $\mathbb{Z}_{10} \backslash B(\mathbb{Z}_{10}) = \{1,2,3,4,6,7,8,9\}$. So, a non-braid graph $\Gamma_{\mathbb{Z}_{10}}$ is illustrated by Figure 3.2.

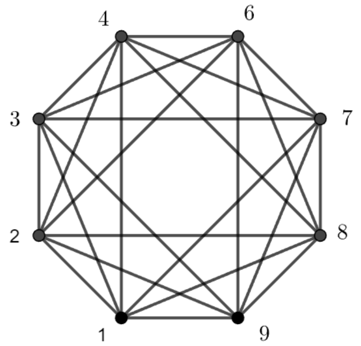

Figure 3.2 $\Gamma_{\mathbb{Z}_{10}}$

From Figure 3.2, we get $\deg(1) = 6 = \deg(2) = \deg(3) = \deg(4) = \deg(6) = \deg(7) = \deg(8) = \deg(9)$. Then,

$$|E(\Gamma_{\mathbb{Z}_5})| = \frac{1}{2} \sum_{x \in \mathbb{Z}_5 \backslash B(\mathbb{Z}_5)} \deg(x)$$

$$= \frac{\deg(1) + \deg(2) + \deg(3) + \deg(4) + \deg(6) + \deg(7) + \deg(8) + \deg(9)}{2}$$

$$= \frac{6+6+6+6+6+6+6+6}{2} = 24.$$

The general zeroth-order Randic index of $\Gamma_{\mathbb{Z}_{10}}$ for $\alpha = 2$ by manual calculations as follows.

$$R_2^0 = \sum_{x \in \mathbb{Z}_{10} \backslash B(\mathbb{Z}_{10})} (\deg(x))^2$$

$$= (\deg(1))^2 + (\deg(2))^2 + (\deg(3))^2 + (\deg(4))^2 + (\deg(6))^2 + (\deg(7))^2 + (\deg(8))^2 + (\deg(9))^2$$

$$= 6^2 + 6^2 + 6^2 + 6^2 + 6^2 + 6^2 + 6^2 + 6^2 = 288.$$

Conclusion

Based on this research, the general formulas of the general zeroth-order Randić index of the non-braid graph for the ring of integers modulo p and $2p$ have been constructed for the cases $\alpha = 1{,}2$ and 3. Additionally, the results include the degree of each vertex and the number of edges obtained.

Acknowledgement

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