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Explicit Exponential Finite Difference Method and Explicit Forward Time Centered Space Scheme for The Newell-Whitehead-Segel Equation

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Abstract
This study numerically investigates the Newell-Whitehead Segel (NWS) equation using the explicit exponential finite difference method and the explicit Forward Time Centered Space (FTCS) scheme. The primary objective is to evaluate the accuracy of the numerical solutions obtained from both methods by comparing them with the analytical solutions. Both methods employ a forward difference for temporal derivatives and a centered difference for spatial derivatives. The discrete equations are algebraically manipulated to approximate the solution of the Newell-Whitehead Segel equation, with MATLAB software used for the calculations. This study also examines the accuracy of both methods and the impact of varying the time step. Results indicate that the explicit exponential finite difference method is more accurate than the explicit FTCS scheme, with errors decreasing as the time step size is reduced. Overall, this study successfully solves the Newell-Whitehead Segel equation numerically, producing results that closely align with analytical solutions.
Keywords: Newell-Whitehead Segel equation; explicit exponential finite difference method; explicit Forward Time Centered Space (FTCS) scheme

1. Introduction
The partial differential equation (PDE) describes the relationship between a multivariable function and its partial derivatives, playing a crucial role in science and engineering. Given the significant effort required to analytically solve nonlinear PDEs for exact solutions, numerical approaches are essential for approximating solutions, reducing computational cost and time. The Newell-Whitehead-Segel equation, one of the remarkable nonlinear PDE is significant in diverse fields, including biology, chemistry, physics, and engineering. A notable application is its role in describing the dynamical behaviour near the bifurcation point of Rayleigh-Benard convection in binary fluid mixtures [1]. It is selected for this study because it is a fundamental problem in various fields. Additionally, it can be addressed using both analytical and numerical methods, facilitating the evaluation of the accuracy of the numerical approaches.

Explicit exponential finite difference method and explicit (FTCS) scheme are utilized in this paper to approximate the exact solutions of the Newel-Whitehead-Segel equation. The finite difference method (FDM) is a widely recognized numerical technique for solving ordinary and partial differential equations. The Finite Difference Method (FDM) has been extensively employed in solving various problems across different fields, utilizing multiple schemes to approximate differential equations such as forward difference, central difference, Crank-Nicolson Scheme, explicit exponential finite difference method and explicit forward time centered space (FTCS). Among these, the explicit exponential finite difference method and the explicit forward time centered space (FTCS) scheme are particularly notable. The explicit exponential finite difference has been applied in mathematical biology. Moreover, the project aims to validate these numerical solutions against established analytical results, and thereby demonstrating the reliability and applicability of the computational approaches in understanding how systems behave under the influence of the NWS equation.

2. Literature Review

2.1. The Newell-Whitehead-Segel (NWS) equation and its application

Numerous physical phenomena are modeled using nonlinear partial differential equations [5–8]. Among these is the Newell-Whitehead-Segel (NWS) equation, a reaction-diffusion equation. It has been widely applied to model and understand phenomena of interest in scientific projects. It is used in heat transfer studies of both homogeneous and heterogeneous materials [9] to analyze temperature distribution in solids.

The general type of Newell-Whitehead-Segel equation

$$u_t = \varepsilon u_{xx} + au + bu^q, \qquad (x,t) \in [A,B] \times [0,T]$$

with boundary conditions

$$u(A,t) = f_1(x) \text{ and } u(B,t) = f_2(x) \;, \qquad \mathrm{t} \geq 0$$

and initial condition

$$u(x,0) = g(x) \;,$$

where a, b and ε are real numbers and q is a positive integer. Additionally, the NWS equation has been employed to describe pattern formation and flow instabilities in fluid dynamics [10]. In biology, it plays a significant role in explaining chemotaxis [11], the response of an organism or cell to chemical stimuli. The creation of striped patterns in two-dimensional biological systems [12], such as zebra skin, the human visual cortex, and fingerprints, has also been modelled by the NWS equation. It aids in understanding plasma structure formation in the field of plasma physics [13]. Furthermore, the NWS equation has been used in ecology [14] to model pattern formation in ecological systems, providing insights into how groups respond to environmental changes. Lastly, the NWS equation is a valuable tool in chemical and materials science, where it is utilized to model and understand the emergence of spatial patterns in chemical reactions and material systems [15].

2.2. The Explicit Exponential Finite Difference Method and Explicit Forward Time Centered Space Scheme

The exponential finite-difference method is an effective technique for solving the Korteweg-de Vries equation at small times, with close agreement to exact solutions [2]. The explicit exponential finite difference methods also solved generalized Huxley and Burgers-Huxley equations, providing accurate numerical solutions [3]. The FTCS scheme, is commonly used for solving the heat equation and advection-diffusion problems. FTCS method is worked very well for the non-local diffusion problem because of its fourth-order accuracy [4]. This project aims to explore and implement numerical methods to solve the NWS equation. Explicit exponential finite difference method and the explicit Forward Time Centered Space (FTCS) scheme will be used.

3. Methodology

3.1. Numerical Discretization using Exponential Finite Difference Scheme

Considering the general type of Newell-Whitehead-Segel equation:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + au + bu^q \tag{3.1}$$

where *a, b* and ε are real numbers and *q* is a positive integer. Consider regular partitions of the intervals consisting of *N* and *K* subintervals, respectively, denoted by $\Delta\, x$ and $\Delta\, t$.

By dividing the equation (3.1) with *u*, then it fields:

$$\frac{\partial}{\partial t}\ln(u)=\frac{\varepsilon}{u}\frac{\partial^2 u}{\partial x^2}+a+\frac{bu^q}{u} \quad . \tag{3.2}$$

For each $i\in\{0,1,\ldots,N\}$ and $j\in\{0,1,\ldots,K\}$, we define $x_i=i\Delta x$ and $t_j=j\Delta t$, and let u_i^j represent an approximation to the exact solution at the point $\left(x_i,t_j\right)$. The difference operators are used in the followings:

$$\frac{\partial}{\partial t}(u_i^j)\approx\frac{u_i^{j+1}-u_i^j}{\Delta t} \quad , \tag{3.3}$$

$$\frac{\partial^2}{\partial x^2}(u_i^j)\approx\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2} \quad . \tag{3.4}$$

for each $i\in\{1,2,\ldots,N-1\}$ and $j\in\{0,1,\ldots,K-1\}$. Obviously, equation (3.3) yields a consistent approximation of order Δt to the exact value of u_t at $\left(x_i,t_j\right)$. Meanwhile equation (3.4) is a consistent approximation of order $(\Delta x)^2$ of $u_{xx}\left(x_i,t_j\right)$. The temporal derivative is discretized with forward difference while the spatial derivative is discretized with centered difference.

Let $\Delta x=h$, and $\Delta t=k$, hence the explicit discrete operator becomes:

$$\Delta u_i^j=\frac{\ln(u_i^{j+1})-\ln(u_i^j)}{k} \tag{3.5}$$

$$\Delta u_i^j=\frac{1}{k}\ln\left(\frac{u_i^{j+1}}{u_i^j}\right) \tag{3.6}$$

From equation (3.2), a finite difference discretization is provided by the difference equations. Such perspective yields the new discrete equations:

$$\Delta u_i^j=\frac{\varepsilon}{u_i^j}\left(\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}\right)+a+b(u_i^j)^{q-1} \tag{3.7}$$

By substituting the equation of explicit discrete operator from equation (3.6) into equation (3.7), then,

$$\frac{1}{k}\ln\left(\frac{u_i^{j+1}}{u_i^j}\right)=\frac{\varepsilon}{u_i^j}\left(\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}\right)+a+b(u_i^j)^{q-1} \tag{3.8}$$

Then, moving *k* to the right-side of equations, equation (3.8) becomes,

$$\ln\left(\frac{u_i^{j+1}}{u_i^j}\right)=\frac{k\varepsilon}{u_i^j}\left(\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}\right)+ka+kb(u_i^j)^{q-1} \tag{3.9}$$

After some algebraic simplification and workings, one may readily that equation (3.9) is equivalent to the explicit equation (3.10),

$$\frac{u_i^{j+1}}{u_i^j}=\exp\left(\frac{k\varepsilon}{u_i^j}\left(\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}\right)+ka+kb(u_i^j)^{q-1}\right)$$

$$u_i^{j+1} = u_i^j \exp\left(\frac{k\varepsilon}{u_i^j}\left(\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} \right) + ka + kb(u_i^j)^{q-1} \right) \tag{3.10}$$

for each $i \in \{1, 2, \ldots, N-1\}$ and $j \in \{0, 1, \ldots, K-1\}$. The computational constants are employed by letting,

$$r = \frac{k\varepsilon}{(\Delta x)^2} \ ,$$

then, equation (3.10) can be written as,

$$u_i^{j+1} = u_i^j \exp\left(r\left(\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{u_i^j} \right) + ka + kb(u_i^j)^{q-1} \right) \tag{3.11}$$

The equation (3.11) is used to approximate the solution for Newell-Whitehead-Segel-type equation.

3.2. Numerical discretization using Explicit scheme (FTCS)

Consider the general type of Newell-Whitehead-Segel equation (3.1) and regular partitions of the intervals consisting of *N* and *K* subintervals with step size denoted by Δx and Δt respectively. Then the temporal derivative is discretized using forward difference formula while spatial derivative is discretized using centered difference formula which are as following,

$$\frac{\partial}{\partial t}(u_i^j) \approx \frac{u_i^{j+1} - u_i^j}{\Delta t} \tag{3.12}$$

$$\frac{\partial^2}{\partial x^2}(u_i^j) \approx \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} \tag{3.13}$$

Hence, by replacing equation (3.12) and equation (3.13) into equation (3.1), we obtain:

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \varepsilon \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\Delta x^2} + au_i^j + b(u_i^j)^q \tag{3.14}$$

After some algebraic simplification and rearrangement, one may readily that equation (3.14) is equivalent to the explicit equation,

$$u_i^{j+1} = \frac{\varepsilon \Delta t (u_{i+1}^j - 2u_i^j + u_{i-1}^j)}{\Delta x^2} + u_i^j(a\Delta t + 1) + b(u_i^j)^q \Delta t \tag{3.15}$$

A constant is employed by letting,

$$r = \frac{\varepsilon \Delta t}{\Delta x^2}$$

then, equation (3.15) can be written as,

$$u_i^{j+1} = r(u_{i+1}^j + u_{i-1}^j) + u_i^j(a\Delta t + 1 - 2r) + b(u_i^j)^q \Delta t \tag{3.16}$$

The equation (3.16) is used to approximate the solution for Newell-Whitehead-Segel-type equation using explicit scheme (FTCS).

4. Results and discussion

4.1. Parameter setting

The experiment is considered the The Newell-Whitehead-Segel (NWS) equation (3.1) with

$$a=1,\ b=-1,\ \varepsilon=1, \text{ and } q=3$$

Given the initial condition,

$$u(x,0)=-0.5+0.5\tanh(0.3536x)$$

and boundary conditions,

$$u(0,t)=-0.5$$

$$\text{and } u(1,t)=-0.5+0.5\tanh(0.3536)$$

with exact the solution is given as,

$$u(x,t)=-0.5+0.5\tanh(0.3536x-0.75t)\ .$$

4.2. Numerical Results based on Explicit Exponential Finite Difference Method

The numerical solutions based on the Explicit Exponential FDM at chosen time step being compared to the exact solutions. This experiment is conducted using MATLAB to get the numerical and exact solutions. Microsoft Excel is also used to construct the comparison table between numerical and exact solutions.

Table 4.2 Explicit Exponential FDM Vs Exact solutions with $\Delta x=0.02$ and $\Delta t=0.0002$ at $t=0.01$

Values of x	Numerical solutions	Exact solutions
0	-0.500000000000000	-0.503749929689082
0.02	-0.497238993855203	-0.500213999986933
0.04	-0.494345079204045	-0.496678048879860
0.06	-0.491334718326710	-0.493142430029747
0.08	-0.488224396380477	-0.489607496965564
0.10	-0.485028807928472	-0.486073602942001
0.12	-0.481762669159590	-0.482541100798337
0.14	-0.478438125686200	-0.479010342817658
0.16	-0.475067342686838	-0.475481680586556
0.18	-0.471659624567552	-0.471955464855407
0.20	-0.468224292336734	-0.468432045399364
0.22	-0.464767965736561	-0.464911770880177
0.24	-0.461297277411494	-0.461394988708945
0.26	-0.457816620896919	-0.457882044909932
0.28	-0.454330398659942	-0.454373283985542
0.30	-0.450841357470404	-0.450869048782579
0.32	-0.447352249344598	-0.447369680359889
0.34	-0.443864724890461	-0.443875517857498
0.36	-0.440380436965057	-0.440386898367348
0.38	-0.436900376063839	-0.436904156805738
0.40	-0.433425532760218	-0.433427625787560
0.42	-0.429956535891936	-0.429957635502445
0.44	-0.426494012945454	-0.426494513592893
0.46	-0.423038409419160	-0.423038585034499

0.48	-0.419590168583097	-0.419590172018349
0.50	-0.416149642604294	-0.416149593835680
0.52	-0.412717180909757	-0.412717166764889
0.54	-0.409293070722368	-0.409293203960966
0.56	-0.405877596257564	-0.405878015347428
0.58	-0.402470957817090	-0.402471907510832
0.60	-0.399073352636955	-0.399075183597930
0.62	-0.395684814736523	-0.395688143215542
0.64	-0.392305375265768	-0.392311082333198
0.66	-0.388934743021519	-0.388944293188618
0.68	-0.385572624486460	-0.385588064196077
0.70	-0.382218137150268	-0.382242679857720

0.72	-0.378870397227843	-0.378908420677854
0.74	-0.375527542554524	-0.375585563080286
0.76	-0.372187711313952	-0.372274379328723
0.78	-0.368847571909683	-0.368975137450292
0.80	-0.365503795349239	-0.365688101162193
0.82	-0.362151066004021	-0.362413529801535
0.84	-0.358784073781484	-0.359151678258359
0.86	-0.355395113261808	-0.355902796911883
0.88	-0.351976488345592	-0.352667131569985
0.90	-0.348517968291999	-0.349444923411927
0.92	-0.345009336773905	-0.346236408934350
0.94	-0.341438103677994	-0.343041819900524
0.96	-0.337791800412796	-0.339861383292870
0.98	-0.334056381520781	-0.336695321268753
1	-0.330217832767732	-0.333543851119530

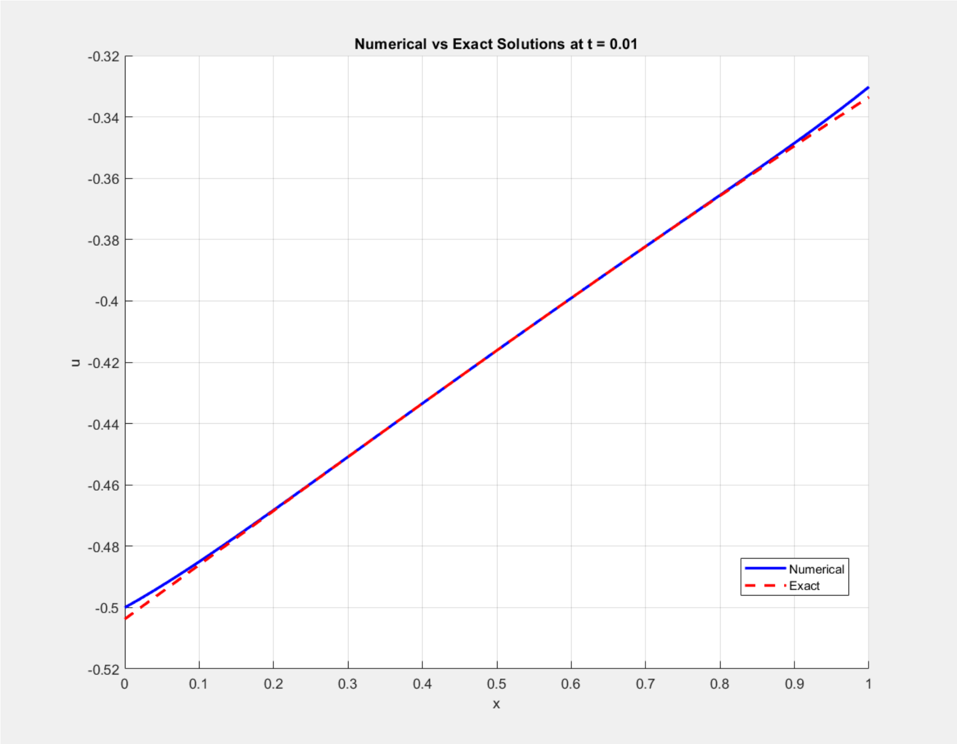

Figure 4.2 Explicit Exponential FDM Vs Exact solutions with $\Delta x = 0.02$ and $\Delta t = 0.0002$ at $t = 0.01$

The Table 4.2 shows the numerical solutions vs exact solutions at $\Delta x = 0.2$ and $\Delta x = 0.02$ respectively. The results are compared with exact solutions which is given by $u(x,t) = -0.5 + 0.5\tanh(0.356x - 0.75t)$. The value of time step is fix at $\Delta t = 0.0002$. The tabulated values are also illustrated in the form of graph as shown in Figure 4.2. It can be seen that the numerical solutions are found to be close to the exact solutions. Most of the numerical values are close to the exact values up to three or four decimal places. At $x = 0$, the numerical solution is −0.5000, while the exact solution is −0.5037. This indicates there is a small difference of 0.0037 between the numerical and exact solutions at this point. At $x = 0.4$ the difference between the numerical and exact solutions is 0.00003, which is relatively smaller compared to other points. This suggests that the numerical solution is closer to the exact solution at the point $x = 0.4$ compared to other points. Overall, the numerical solution exhibits some difference but still in a close gap to exact solutions.

4.3. Explicit scheme (FTCS) for the solution of Newell-Whitehead-Segel equation

The explicit scheme (FTCS) is employed to solve the Newell-Whitehead-Segel (NWS) equation in the same problem as the previous section. The numerical solutions of the equations were computed using MATLAB software by setting up the step size with $\Delta x = 0.02$ and $\Delta t = 0.0002$ at $t = 0.01$. The results were then tabulated and compared with the exact solutions in Table 4.3 below. Additionally, the approximation of the solutions was visualized using a graph plot to investigate the accuracy of the method used, as shown in Figure 4.3 below.

Table 4.3 FTCS vs Exact solutions using explicit scheme with $\Delta x = 0.02$ and $\Delta t = 0.0002$ at $t = 0.01$

Values of x	Numerical	Exact
0	-0.500000000000000	-0.503749929689082
0.02	-0.497238969735857	-0.500213999986933
0.04	-0.494345029775404	-0.496678048879860
0.06	-0.491334644795495	-0.493142430029747
0.08	-0.488224296077026	-0.489607496965564
0.10	-0.485028684000438	-0.486073602942001
0.12	-0.481762518537710	-0.482541100798337
0.14	-0.478437953432922	-0.479010342817658
0.16	-0.475067146072945	-0.475481680586556
0.18	-0.471659409810563	-0.471955464855407
0.20	-0.468224057386200	-0.468432045399364
0.22	-0.464767716879940	-0.464911770880177
0.24	-0.461297013303047	-0.461394988708945
0.26	-0.457816346968132	-0.457882044909932
0.28	-0.454330114129169	-0.454373283985542
0.30	-0.450841066432156	-0.450869048782579
0.32	-0.447351951398505	-0.447369680359889
0.34	-0.443864422768057	-0.443875517857498
0.36	-0.440380130491912	-0.440386898367348
0.38	-0.436900066870979	-0.436904156805738
0.4	-0.433425220792881	-0.433427625787560
0.42	-0.429956222035907	-0.429957635502445
0.44	-0.426493697203168	-0.426494513592893
0.46	-0.423038092250059	-0.423038585034499
0.48	-0.419589850014084	-0.419590172018349
0.50	-0.416149322910219	-0.416149593835680
0.52	-0.412716860125579	-0.412717166764889
0.54	-0.409292749127971	-0.409293203960966
0.56	-0.405877273885444	-0.405878015347428
0.58	-0.402470635147228	-0.402471907510832
0.60	-0.399073029685947	-0.399075183597930
0.62	-0.395684492444591	-0.395688143215542
0.64	-0.392305053607178	-0.392311082333198
0.66	-0.388934423751601	-0.388944293188618
0.68	-0.385572307484069	-0.385588064196077
0.70	-0.382217825396484	-0.382242679857720
0.72	-0.378870090409785	-0.378908420677854
0.74	-0.375527245206856	-0.375585563080286
0.76	-0.372187422780645	-0.372274379328723
0.78	-0.368847298300340	-0.368975137450292
0.80	-0.365503535469457	-0.365688101162193
0.82	-0.362150827043302	-0.362413529801535
0.84	-0.358783853829070	-0.359151678258359
0.86	-0.355394919549731	-0.355902796911883
0.88	-0.351976318214981	-0.352667131569985
0.90	-0.348517827732771	-0.349444923411927
0.92	-0.345009222669659	-0.346236408934350
0.94	-0.341438019809738	-0.343041819900524

0.96	-0.337791743962340	-0.339861383292870
0.98	-0.334056353947223	-0.336695321268753
1	-0.330217832767732	-0.333543851119530

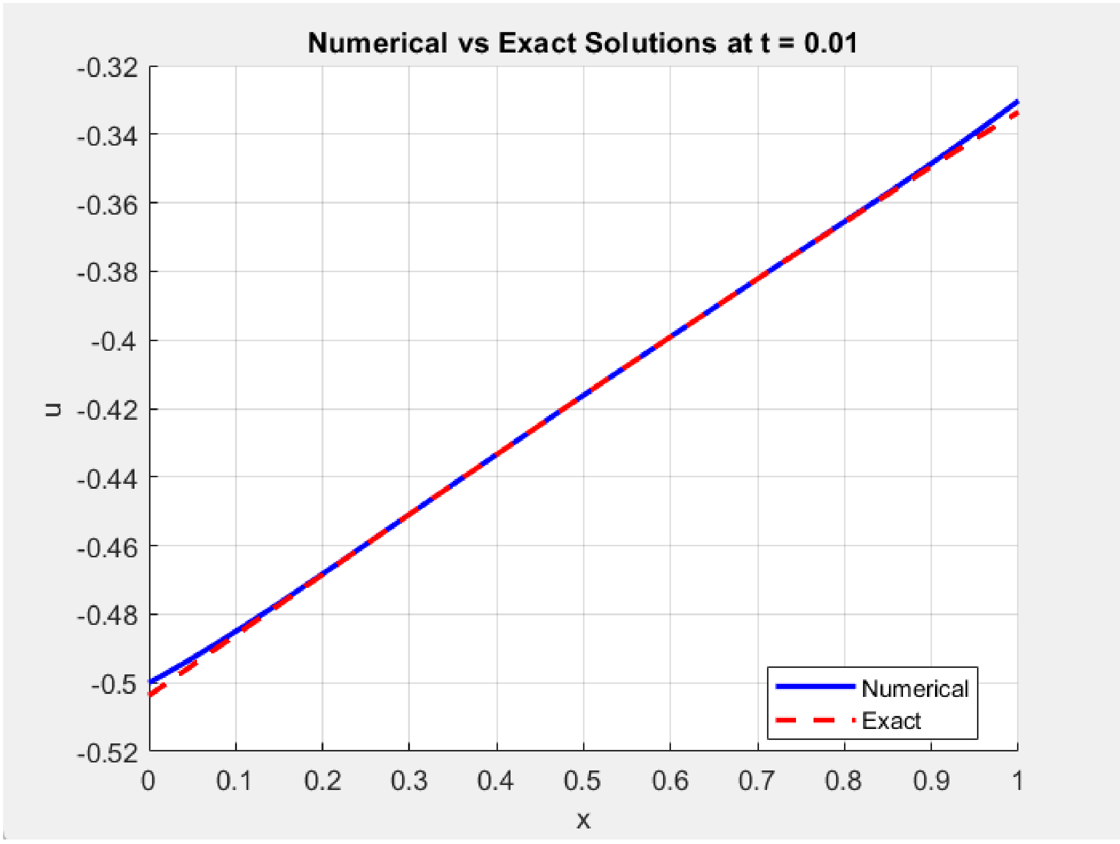

Figure 4.3 Numerical solutions Vs Exact solutions using FTCS scheme with $\Delta x = 0.02$ and $\Delta t = 0.0002$ at $t = 0.01$

In Table 4.3, the numerical results are presented alongside the exact solutions, providing a clear comparison that highlights the effectiveness and accuracy of the explicit scheme (FTCS). Furthermore, the graphical representation in Figure 4.3 offers a visual perspective, enabling a more insightful understanding of how closely the numerical solutions align with the exact solutions.

4.4. Comparison of Explicit Exponential Finite Difference Method and Explicit scheme (FTCS)

Table 4.4 Absolute errors of numerical solutions using exponential and explicit scheme with $\Delta x = 0.02$ and $\Delta t = 0.0002$ at $t = 0.01$

Values of x	Exponential	Explicit
0	0.003749929689082	0.003749929689082
0.02	0.002975006131730	0.002975030251076
0.04	0.002332969675815	0.002333019104455

0.06	0.001807711703037	0.001807785234252
0.08	0.001383100585088	0.001383200888538
0.10	0.001044795013529	0.001044918941563
0.12	0.000778431638746	0.000778582260627
0.14	0.000572217131459	0.000572389384737
0.16	0.000414337899718	0.000414534513611
0.18	0.000295840287855	0.000296055044843
0.20	0.000207753062631	0.000207988013164
0.22	0.000143805143616	0.000144054000237
0.24	0.000097711297451	0.000097975405898
0.26	0.000065424013013	0.000065697941800
0.28	0.000042885325600	0.000043169856373
0.30	0.000027691312175	0.000027982350423
0.32	0.000017431015292	0.000017728961385
0.34	0.000010792967037	0.000011095089441
0.36	0.000006461402292	0.000006767875437
0.38	0.000003780741899	0.000004089934759
0.4	0.000002093027342	0.000002404994678
0.42	0.000001099610509	0.000001413466537
0.44	0.000000500647439	0.000000816389725
0.46	0.000000175615339	0.000000492784440
0.48	0.000000003435252	0.000000322004264
0.50	0.000000048768614	0.000000270925461
0.52	0.000000014144868	0.000000306639310
0.54	0.000000133238597	0.000000454832995
0.56	0.000000419089864	0.000000741461984
0.58	0.000000949693742	0.000001272363603
0.60	0.000001830960975	0.000002153911982
0.62	0.000003328479019	0.000003650770951
0.64	0.000005707067431	0.000006028726020
0.66	0.000009550167099	0.000009869437017
0.68	0.000015439709617	0.000015756712008
0.70	0.000024542707451	0.000024854461235
0.72	0.000038023450012	0.000038330268069
0.74	0.000058020525762	0.000058317873430
0.76	0.000086668014771	0.000086956548078
0.78	0.000127565540608	0.000127839149952
0.80	0.000184305812954	0.000184565692735
0.82	0.000262463797514	0.000262702758232
0.84	0.000367604476875	0.000367824429289
0.86	0.000507683650075	0.000507877362152
0.88	0.000690643224393	0.000690813355003
0.90	0.000926955119928	0.000927095679156
0.92	0.001227072160446	0.001227186264691
0.94	0.001603716222530	0.001603800090786
0.96	0.002069582880074	0.002069639330531
0.98	0.002638939747972	0.002638967321530
1	0.003326018351799	0.003326018351799
Average	0.000591356379920	0.000591583119595

Table 4.4 shows the absolute errors of numerical solutions using exponential and explicit schemes for various values of x at $t = 0.01$ with $\Delta x = 0.02$ and $\Delta t = 0.01$, illustrating their convergence and accuracy. Both methods yield very close results across all x values, with differences in the absolute errors being minimal and nearly indistinguishable, particularly as x approaches the value of $x = 1$. The exponential and explicit schemes start with identical values at $x = 0$, and although slight deviations occur at other points, these divergence are minor. For instance, at $x = 0.5$, the absolute values are 0.000000048768614 for the exponential scheme and 0.000000270925461 for the explicit scheme. On average, the absolute errors across all x values are 0.000591356379920 for the exponential scheme and 0.000591583119595 for the explicit scheme, indicating that both methods perform with high precision and reliability. This close alignment underscores the effectiveness of both schemes in providing accurate numerical solutions for the given parameters. On the other hand, average absolute errors for exponential scheme are slightly lesser compare to explicit scheme indicating that exponential are more accurate to the exact solutions.

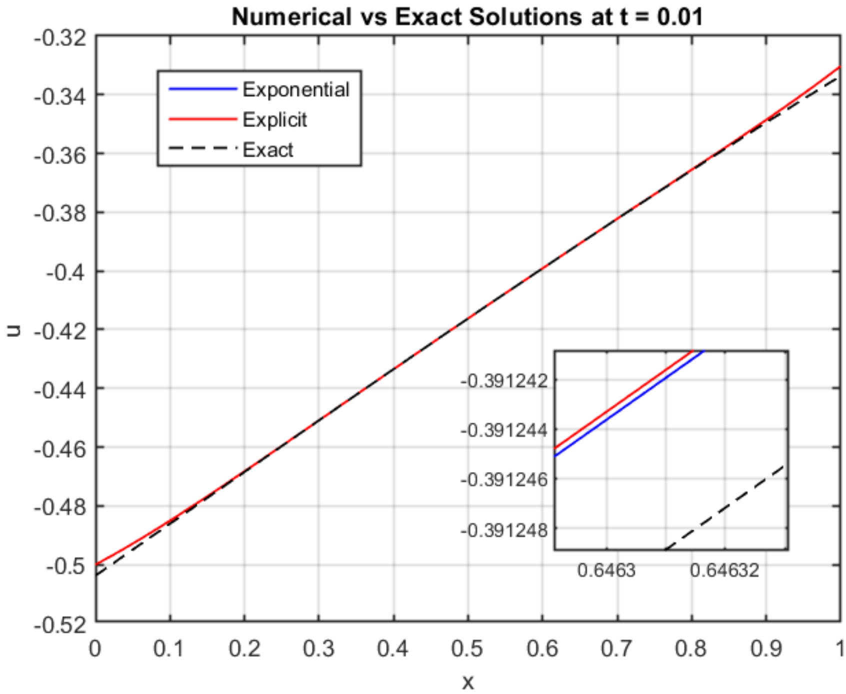

Figure 4.4 Numerical solutions for exponential and explicit scheme vs exact solutions with $\Delta x = 0.02$ and $\Delta t = 0.0002$ at $t = 0.01$

From the Figure 4.4, it visual the approximation of numerical to the exact solutions for exponential and explicit scheme. In general, both numerical solutions are close to exact solutions and slightly diverge at initial and boundary conditions. Figure 4.7 indicates that exponential scheme solutions getting more closer to the exact solutions compare to explicit scheme. Hence, this result shows that exponential is more accurate than explicit scheme.

Conclusion

The numerical results are presented alongside the exact solutions, providing a clear comparison that highlights the effectiveness and accuracy of both the explicit exponential finite difference method and the explicit scheme (FTCS). The graphical representation offers a visual perspective, enabling a deeper understanding of how closely the numerical solutions align with the exact solutions. The comparison reveals that the average absolute errors for the exponential scheme are slightly lower than those for the explicit scheme, indicating greater accuracy. This finding is further supported by the closer approximation of numerical solutions to exact solutions in the exponential scheme compared to the explicit scheme. Thus, the results demonstrate that the exponential scheme is more accurate.

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