Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 23, 2024, page 108-118

Numerical Solution of One-dimensional Shallow Water Equations using MacCormmack Method

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Abstract
The one-dimensional shallow water wave equation is a fundamental equation in fluid dynamics and is widely used to model various practical problems. One of the particular problems addressed in this study is ocean wave dynamics, and the model that is focused on is the pond model. Therefore, the discussion focuses on solving the one-dimensional shallow water wave equation. The research methodology involved a literature review of the shallow water wave equation and its solution methods. The one-dimensional shallow water wave is derived from principles of conservation of mass and conservation of momentum. This study used the MacCormack method, a finite difference technique consisting of two steps, namely the predictor and corrector steps. The predictor step used forward differencing to estimate the solution at the next time step, while the corrector step used backward differencing to refine this estimate. These two-step processes help to improve the accuracy of the solution. The MacCormack method is employed to conduct numerical simulations of the pond model for one-dimensional shallow water wave equations over flat topography. The simulation results indicated that the channel for flat topography was obtained for the height of water surface elevation and the velocity of the water at t = 0, 20, 40, and 60 seconds. Based on the simulation results, the water surface for flat topography moves symmetrically and the amplitude and speed of the waves on the surface of the water will always change.

Keywords: One-dimensional; Shallow water wave; MacCormack method; Pond model;

1. Introduction

The wave equation is a fundamental partial differential equation (PDE) that describes the propagation of various types of waves, such as sound waves, light waves, and water waves. It plays a crucial role in fluid dynamics, offering insights into the behaviour of waves under different conditions and in various media. Exact solutions to the wave equation provide valuable understanding and predictive power for phenomena observed in nature and engineering applications [2].

In this study, the focus is on one specific model of water waves known as shallow water waves. This model is particularly important because it describes the behaviour of waves in environments where the water depth is significantly less than the wavelength of the wave. Shallow water waves are essential for applications such as predicting tsunami behaviour, managing flood risks, and analyzing ripples in pond water. The ability to accurately model and predict the behaviour of shallow water waves has significant implications for coastal engineering, environmental management, and disaster preparedness
.

The one-dimensional shallow water wave equation is a simplified form of the general shallow water equations, derived from the principles of conservation of mass and momentum [4],[5]. It is used to model the flow of shallow water in a single spatial dimension, providing a more tractable problem for numerical analysis [6],[7]. However, solving this equation analytically can be challenging, especially when dealing with complex boundary conditions and non-flat topographies. Therefore, numerical methods, such as the MacCormack method, are employed to obtain approximate solutions.

The MacCormack method, introduced by Robert W. MacCormack in 1969, is a finite difference technique that uses a predictor-corrector approach to solve time-dependent partial differential equations This method is known for its simplicity and accuracy, making it a popular choice in computational fluid dynamics. The predictor step estimates the solution at the next time step using forward differencing, while the corrector step refines this estimate using backward differencing [15]. This two-step process helps to improve the accuracy of the numerical solution.

The primary goal of this study is to analyze the one-dimensional shallow water wave equation using the MacCormack method. The research aims to develop and implement the MacCormack method specifically for this purpose and to assess its accuracy and stability under various conditions. The study will involve deriving the shallow water wave equation, implementing the MacCormack method for numerical simulation, and analyzing the results to gain insights into the behavior of shallow water waves .

By focusing on the one-dimensional shallow water wave equation, this research contributes to the broader understanding of wave dynamics in shallow water environments

2. Mathematical Formulation

2.1 Derivation of Shallow Water Equations

The shallow water wave equation consists of the continuity equation and the momentum equation, which can be derived from the law of conservation of mass and the law of conservation of momentum, respectively.

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial \left(u^2 h\right)}{\partial x} = -\frac{1}{2} g \frac{\partial h^2}{\partial x} \tag{2}$$

Equation (1) and (2) are two equations that need to be solved together. This system of equations is called the one-dimensional (1D) shallow water wave equation with the acceleration due to gravity g, dependent variables u and h as well as independent variables x and t. Based on [17], it is given a system of equations which is equivalent to the following equations (1) and (2).

$$\frac{\partial \eta}{\partial t} + \frac{\partial \left(u(D+\eta)\right)}{\partial x} = 0 \tag{3}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x} \tag{4}$$

where:

- η is the elevation of the water surface,
- D is a function of the water depth
- $D + \eta$ is the total water depth
- g is acceleration due to gravity

Equation (3) represents the conservation of mass (continuity equation), and equation (4) represents the conservation of momentum (motion equation). The model can be illustrated in Figure 1.

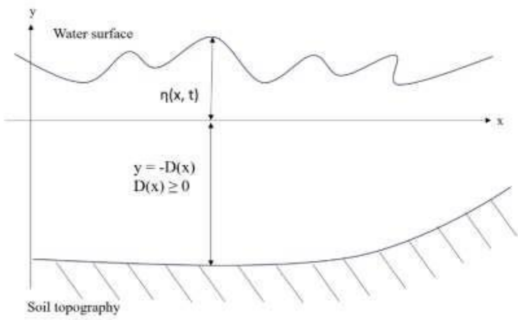


Figure 1: Illustration of the equation (3) and (4)

3. Numerical Computation

3.1 Derivation 1D Shallow Water Equation Using MacCormack Method

By using equations (3) and (4) the derivation of the shallow water wave equation using the MacCormack method is as follows. Note that the approximation formula for the MacCormack method for the shallow water wave equation was defined:

$$\Delta x = \frac{L}{N_x}, \qquad x_i = x_0 + i\Delta x \; ; \; i = 0, 1, 2, \ldots, N_x$$

$$\Delta t = \frac{L}{N_t}, \qquad x_i = n\Delta t \; ; \; n = 0, 1, 2, \ldots, N_{t,}$$

Where:

- L is the flow length
- t is the time duration
- N_x is number of discretization of space variable

- N_t is number of time variable discretization

First, the predictor equation for the MacCormack method is derived from system equations (3) and (4). The approximation of the partial derivatives of the space and time variables in the system equations is achieved using a first-order forward finite difference scheme. By applying a one-step forward difference scheme, the partial derivatives of the space and time variables.

$$\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}+\frac{u_{i+1}^n\left(D_{i+1}+\eta_{i+1}^n\right)-u_i^n\left(D_i+\eta_i^n\right)}{\Delta x}=0$$
$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+u_i^n\left(\frac{u_{i+1}^n-u_i^n}{\Delta x}\right)=-g\left(\frac{\eta_i^{n+1}-\eta_i^n}{\Delta x}\right) \tag{5}$$

In the predictor step, the value of η_i^{n+1} and u_i^{n+1} is a temporary value at the time level n+1, by using the notation $\eta_i^{\overline{n+1}}$ and $u_i^{\overline{n+1}}$. Equation (5) can be written as:

$$\eta_i^{\overline{n+1}}=\eta_i^n-\alpha\left(u_{i+1}^n\left(D_{i+1}+\eta_{i+1}^n\right)-u_i^n\left(D_i+\eta_i^n\right)\right)$$
$$u_i^{\overline{n+1}}=u_i^n-\alpha u_i^n\left(u_{i+1}^n-u_i^n\right)-g\alpha\left(\eta_{i+1}^n-\eta_i^n\right) \tag{6}$$

where, $\alpha=\frac{\Delta t}{\Delta x}$

In corrector step, the MacCormack method is derived from the system equations. The partial derivatives of the space and time variables are approximated using a first-order backward finite difference scheme. The partial derivatives of the space and time variables in the system equation are then approximated using a one-step backward difference scheme and a half-step backward difference scheme. The equation can be written as follows:

$$\frac{\eta_i^{n+1}-\eta_i^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}+\frac{u_i^{n+1}\left(D_i+\eta_i^{n+1}\right)-u_{i-1}^{n+1}\left(D_{i-1}+\eta_{i-1}^{n+1}\right)}{\Delta x}=0$$
$$\frac{u_i^{n+1}-u_i^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}+u_i^{n+1}\left(\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}\right)=-g\left(\frac{\eta_i^{n+1}-\eta_{i-1}^{n+1}}{\Delta x}\right) \tag{7}$$

The value of $\eta_i^{\overline{n+1}}$ and $u_i^{\overline{n+1}}$ were used from the predictor step to replace the input value of Equation (7), so it can be written as:

$$\eta_i^{n+1} = \eta_i^{n+\frac{1}{2}} - \frac{\alpha}{2}\left(\overline{u_i^{n+1}}\left(D_i + \overline{\eta_i^{n+1}} \right) - \overline{u_{i-1}^{n+1}}\left(D_{i-1} + \overline{\eta_{i-1}^{n+1}} \right) \right)$$

$$u_i^{n+1} = u_i^{n+\frac{1}{2}} - \frac{\alpha \overline{u_i^{n+1}}}{2}\left(\overline{u_i^{n+1}} - \overline{u_{i-1}^{n+1}} \right) - \frac{g\alpha}{2}\left(\overline{\eta_i^{n+1}} - \overline{\eta_{i-1}^{n+1}} \right) \tag{8}$$

The value of $\eta_i^{n+\frac{1}{2}}$ and $u_i^{n+\frac{1}{2}}$ were replaced by the average value of η and u at the time level n with a temporary value at the time level n + 1, so that Equation (8) becomes:

$$\eta_i^{n+1} = \frac{1}{2}\left(\eta_i^n + \overline{\eta_i^{n+1}} - \alpha\left(\overline{u_i^{n+1}}\left(D_i + \overline{\eta_i^{n+1}} \right) - \overline{u_{i-1}^{n+1}}\left(D_{i-1} + \overline{\eta_{i-1}^{n+1}} \right) \right) \right)$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + \overline{u_i^{n+1}} - \alpha \overline{u_i^{n+1}}\left(\overline{u_i^{n+1}} - \overline{u_{i-1}^{n+1}} \right) - g\alpha\left(\overline{\eta_i^{n+1}} - \overline{\eta_{i-1}^{n+1}} \right) \right) \tag{9}$$

Where, $\alpha = \frac{\Delta t}{\Delta x}$ .Equation (9) is the corretor step of the MacCormack method for Shallow Water Equation.

4. Results and Discussion

By approach to Equation (6) and Equation (9) were used to solve the problem with the help of MATLAB software to get the velocity and amplitude at the exact time. The problem to be discussed is the ripples in the pond water

4.1 Pond model

This is a simulation of a one-dimensional (1D) pond model that represents shallow water waves. Initially, the water surface is motionless or has zero velocity. Subsequently, it undergoes disruption in the form of primary waves. In this model, the water waves that come into contact with the pond wall will consistently undergo reflection. Consequently, reflecting boundary conditions were employed as the outcome. Assume that the pond has a length of L meters (m). The computational domain for the variable space x was [0,L]. The first wave occurred at the centre of the pond. Put simply, the highest point of the first wave occurred when x = 10. Space limit x=0 and x=L represents the pond wall. Meanwhile, the time used for the simulation is t (in seconds(s)) with the computational domain for the time variable t is [0, T] with acceleration due to gravity g = 9.8 m/s.

4.1.1 Flat Topography

The pond has a water depth below the *x-axis* (equilibrium level) following function:

$$D(x) = 0.08042, 0 \leq x \leq 20.$$

The initial conditions for η and u was in the form of $\eta(x,0) = 0.05e^{-x2}$ and $u(x,0) = 0$ for each $0 \leq x \leq 20$. The movement of water waves in the pond will be observed for 60 s using the reflective boundary conditions. With $N_x = 200$ and $N_t = 3000$.

The numerical calculations were carried out using the MacCormack method with equations (6) and (9). The results of the numerical simulations of the pond model in this experiment at t = 0,20,40,60 are presented in Figures 2,3,4,5

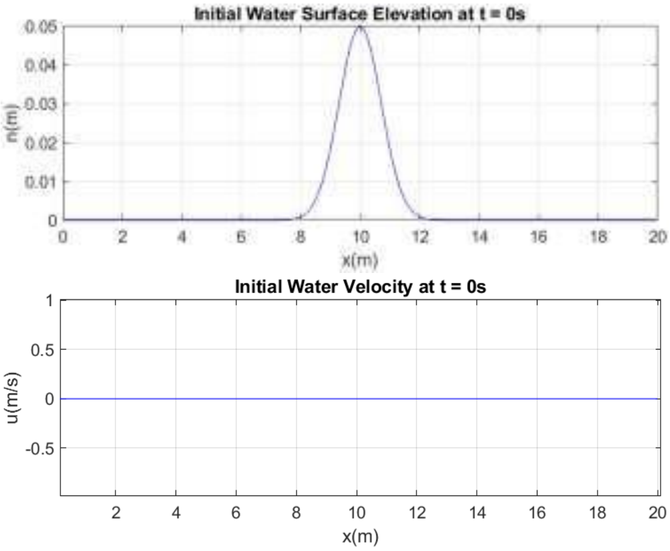
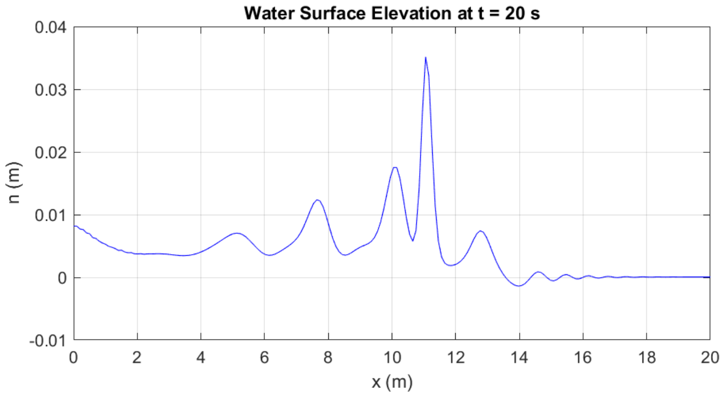


Figure 2: Water surface elevation (Top) and Water velocity (Bottom) at t = 0 s

At the initial time t = 0s, the water surface is calm with no wave activity observed. The water surface elevation is at equilibrium, showing a flat line indicating no initial disturbance. This equilibrium state is essential as it sets the baseline for the numerical simulation. The initial condition of zero disturbance ensures that any subsequent changes in the water surface elevation and velocity can be directly attributed to the numerical simulation's application. Correspondingly, the water velocity is zero across the entire domain, confirming the initial state of rest.

This state of rest is crucial for observing the effects of the applied forces and boundary conditions as the simulation progresses. The initial calmness represents a controlled environment where variables can be introduced systematically to study their impact. This setup establishes the baseline for observing how the water waves evolve over time under the influence of the MacCormack method for numerical simulation. By starting with a calm surface, the simulation can accurately depict the generation and propagation of waves, providing insights into the dynamics of shallow water waves in a controlled setting.



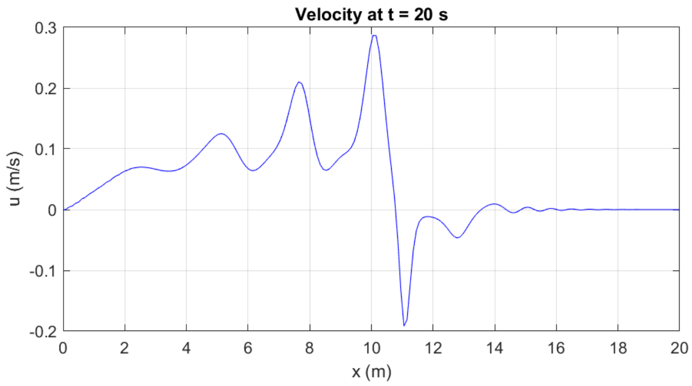
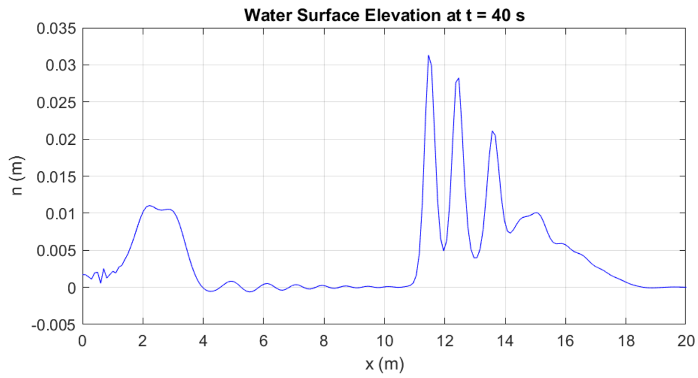


Figure 3: Water surface elevation (Top) and Water velocity (Bottom) at t = 20 s

At t = 20s, the simulation shows the initial wave propagation. The water surface elevation graph indicates a noticeable disturbance, with the highest amplitude reaching approximately 0.03515 m. This initial disturbance is a result of the primary wave generated within the pond, demonstrating the model's ability to simulate wave generation accurately. The formation of this wave can be attributed to the initial conditions set in the simulation and the application of the MacCormack method.

The velocity graph reflects this change, with the maximum velocity reaching 0.28667 m/s The velocity profile indicates the speed at which the water particles are moving, correlating with the observed wave height. The wave has started to reflect off the pond walls, indicating the effectiveness of the reflective boundary conditions applied in the simulation.

This reflection is crucial for studying wave interactions and energy conservation within the pond. As the waves hit the boundaries, they are reflected back, creating interference patterns that are vital for understanding wave dynamics. This phase demonstrates the initial energy distribution across the water surface. The interaction between the generated waves and the boundary conditions highlights the model's capability to simulate realistic scenarios where waves continuously interact with their environment. The data at this stage helps in analyzing the immediate response of the system to the initial disturbance and sets the stage for observing longer-term wave behavior and energy dissipation.



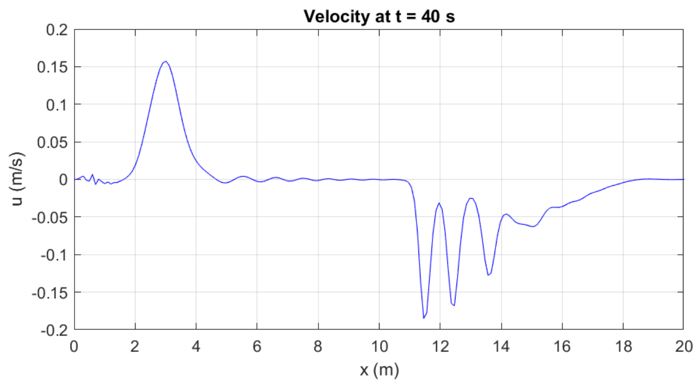
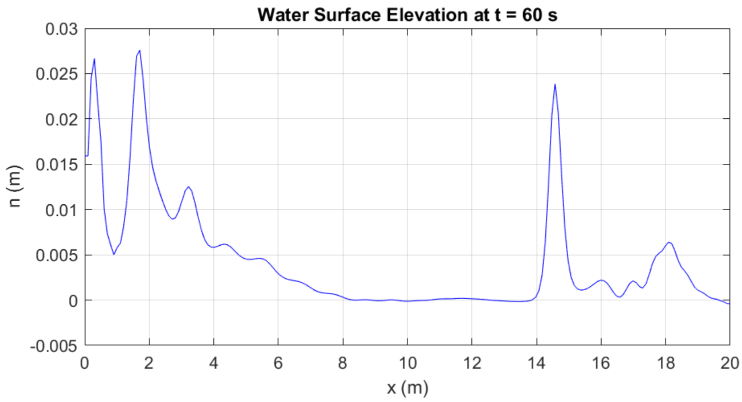


Figure 4: Water surface elevation (Top) and Water velocity (Bottom) at t = 40 s

By t = 40s, the water waves continue to propagate and interact with the pond boundaries. The amplitude of the waves has slightly decreased to 0.03132 m, and the velocity has reduced to 0.18513 m/s. These changes suggest some energy dissipation within the system as the waves reflect off the boundaries and interfere with each other. The decrease in amplitude and velocity indicates that the energy initially imparted to the system is gradually being absorbed and redistributed through interactions with the pond walls and internal frictional forces.

The symmetric pattern of wave movement observed indicates that the flat topography is maintaining consistent wave reflections. This symmetry is essential for validating the model's accuracy in representing real-world scenarios where uniform topography leads to predictable wave behavior. The reflection and interference of waves provide insights into how energy is conserved and transferred within the system. This stage of the simulation helps in understanding the intermediate dynamics of wave propagation and the role of boundary conditions in influencing wave behavior. The results highlight the importance of considering energy dissipation mechanisms in numerical simulations to accurately predict the behavior of shallow water waves over time.



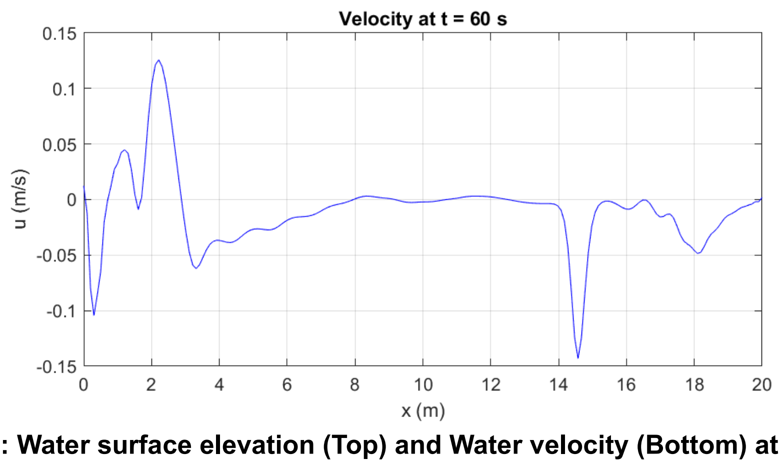


Figure 5: Water surface elevation (Top) and Water velocity (Bottom) at t = 60 s

At the final observation time of t = 60s, the simulation shows a further decrease in wave amplitude to 0.02759 m and a reduction in velocity to 0.14328 m/s. The continuous interaction between the waves and the pond boundaries leads to a gradual loss of energy. This energy loss can be attributed to the reflective boundary conditions and the inherent damping effects within the numerical model. The results illustrate how the system stabilizes over time with decreasing wave energy. The gradual reduction in amplitude and velocity suggests that the system is reaching a state of equilibrium where the energy imparted to the waves is balanced by the energy dissipated through reflections and internal damping. The average amplitude and velocity values, 0.02401 meters and 0.01025 meters per second respectively, highlight the energy dissipation effect over time in a system with reflective boundaries.

This phase demonstrates the long-term behavior of the simulated shallow water waves, providing insights into how wave energy diminishes and stabilizes in a confined environment. The observed trends in wave amplitude and velocity over time are critical for understanding the long-term stability and energy dynamics of shallow water waves. This data is essential for applications such as coastal engineering and environmental studies, where predicting wave behavior over extended periods is crucial. The final results affirm the effectiveness of the MacCormack method in simulating shallow water wave dynamics and provide a foundation for further studies involving more complex topographies and wave interactions.

5. Conclusion

In this article, numerical simulations using the MacCormack method have been carried out for 1D shallow water wave pond model for flat topography. Based on the results of numerical simulations that have been carried out, the water surface for flat topography moves symmetrically and the amplitude and speed of the waves on the surface of the water will always change. As a result, it can be concluded that the topographical shape can affect the movement of the water surface.

Acknowledgment

I extend my deepest gratitude to my supervisor, Dr. Muhamad Najib Zakaria, for his invaluable guidance and support. I also thank my friends and colleagues at Universiti Teknologi Malaysia (UTM) for their constant support, and my family for their unwavering encouragement.

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