Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 23, 2024, page 127-132

The Energy and the Minimum Degree Energy of the Cayley Graph Associated to Certain Subsets of the Dihedral Groups Up to Order Eight

Yew Yun Ru, Nor Haniza Sarmin
Department of Mathematical Sciences, Faculty of Science, 81310 UTM Johor Bahru, Johor, Malaysia
Corresponding author: nhs@utm.my

Abstract
The energy of a simple graph was first inspired by the Hückel Molecular Orbital theory to estimate the energy associated with π-electron orbitals of molecules. In this research, the energy and the minimum degree energy of the Cayley graph associated to the dihedral group of order six and eight with subsets of order one have been computed using some concepts and properties in graph theory, group theory, and linear algebra. The Cayley graph is constructed based on each subset of the dihedral group. Then, the adjacency matrix and the minimum degree matrix are determined to obtain their eigenvalues. Finally, the energy and the minimum degree energy of the Cayley graphs are computed and presented. The results show that the energy and the minimum degree energy of the Cayley graph associated to the dihedral group of order six and eight with subsets of order one are the same. Besides, the energies obtained are all even.
Keywords: Cayley graph; energy of graph; dihedral group; graph theory; group theory

Introduction
The study on Cayley graphs was first initiated by Arthur Cayley in 1878, and since then, numerous researchers have shown their interest in this field. For instance, Adiga and Ariamanesh have specifically studied the Cayley graphs on symmetric groups in 2012 [1]. Furthermore, Ramaswamy and Veena have determined the energy of unitary Cayley graphs [2] which was extended from Balakrishnan in 2004 [3].

Meanwhile, the study on the energy of a graph was first considered by Gutman in 1978 [4]. Gutman defined the energy of a graph as the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph, which was inspired from the Hückel Molecular Orbital Theory (HMO) proposed in the 1930s by Hückel. Chemists have employed the Hückel Molecular Orbital Theory to estimate the energy levels associated with π-electron orbitals within conjugated hydrocarbons.

Adiga and Swamy first described the minimum degree energy of a graph as the total of the absolute values of the eigenvalues of the minimum degree matrix of the graph in 2010 [5]. Subsequently, extensive research has been conducted on the minimum degree energy of certain graphs of groups. For instance, Basavanagoud and Jakkannavar have computed the minimum degree energy of regular graphs and obtain bounds for the largest minimum degree eigenvalue and minimum degree energy [6]. Since then, numerous authors including Rao [7], and Romdhini and Nawawi [8], have expanded the research on the minimum degree energy of certain graph of groups.

In this research, the energy and the minimum degree energy of the Cayley graphs associated to the dihedral groups of order six and eight with subsets of order one are determined. First, the Cayley graphs are constructed with the subset S, then the adjacency matrix and the minimum degree matrix for the graphs are determined, and lastly the eigenvalues of their matrix are obtained which are used to compute the energy and the minimum degree energy of the Cayley graphs.

This paper is structured as follows: in Section 1, previous studies on Cayley graphs, energy and minimum degree energy of a graph are discussed, while in Section 2, the preliminary results that are being used for this study are included. In Section 3, the main results are presented in the form of propositions. Finally, Section 4 gives the conclusion of the main results.

Preliminaries

In this section, some definitions that are used in this research are included.

Definition 1 [9] Dihedral Groups

The dihedral group of order $2n$, denoted by D_{2n}, is the group of symmetries of an n-gon. These symmetries include rotations, denoted by *a*, and reflections, denoted *b.* The group presentation is $D_{2n} = \langle a, b|\ a^n = b^2 = 1\ and\ bab = a^{-1}\rangle$.

Definition 2 [10] Cayley Graph of a Group

Let G be a finite group with identity 1. Let S be a subset of G satisfying $1 \notin S$ and that $S = S^{-1}$; that is, $s \in S$ if and only if $s^{-1} \in S$. The Cayley graph $Cay(G, S)$ on G with subset S is defined as follows:

- the vertices are the elements of G.
- there is an edge joining v_1 and v_2 if and only if $v_2 = sv_1$ for some $s \in S$.

The set of edges is denoted as $E\big(Cay(G, S)\big) = \Big\{\{v_{i,}v_{i,}\}|v_i \text{ is adjacent to } v_j\Big\}$.

Remark: The relation between the two vertices can also be rewritten as $v_2v_1^{-1} = s$ for some *s* ∈ *S.*

Definition 3 [10] Complete Graph

A complete graph K_n has n vertices and each of it is adjacent to all of the others.

Definition 4 [11] Union of Graph

Let G_1 and G_2 be subgraphs of a graph G. The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Note: The union of *m* copies of K_n, that is $\underbrace{K_n \cup K_n \cup \cdots \cup K_n}_{m \text{ copies}}$, is denoted by mK_n.

Definition 5 [12] Adjacency Matrix

Let R be a graph with $V(R) = \{1, 2, \ldots, n\}$ and $E(R) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of R, denoted by $A(R)$, is the $n \times n$ matrix defined as follows: The rows and the columns of $A(R)$ are indexed by $V(R)$. If $i \neq j$ then the (i,j)-entry of $A(R)$ is 0 for vertices i and j non-adjacent, and the (i,j)-entry is 1 for i and j adjacent. The (i,i)-entry of $A(R)$ is 0 for $i = 1, 2, \ldots, n$.

Definition 6 [12] Eigenvalues of a Matrix

The roots of the characteristic equation $det(A - \lambda I) = 0$ of A are called the eigenvalues of A.

Definition 7 [12] Energy of a Graph

Consider *R* to be a graph and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of *R*. The energy of *R*, $\varepsilon(R)$, is defined as follows:

$$\varepsilon(R) = \sum_{i=1}^{n} |\lambda_i|.$$

Definition 8 [5] Minimum Degree Matrix

Let R be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let $d_i = \deg(v_i)$ be the degree of v_i, $i = 1,2, \ldots, n$. The minimum degree matrix of the graph R is defined by $M(R) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} \min\{d_i, d_j\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the minimum degree matrix $M(R)$ is defined by

$$g(R : \lambda) = det\big(\lambda I - M(R)\big).$$

Definition 9 [5] Minimum Degree Energy

Let $M(R)$ be the minimum degree matrix of a graph R and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. Then, the minimum degree energy of the graph R is defined as

$$\varepsilon_M(R) = \sum_{i=1}^{n} |\lambda_i|.$$

Main results

In this section, the main results are specified in terms of propositions. The Cayley graphs associated to the dihedral groups D_6 and D_8 with subsets S of order one are constructed. Then, the energy and the minimum degree energy of the Cayley graphs are obtained.

3.1 The Cayley Graph Associated to the Dihedral Group of Order Six and Eight with Subsets of Order One

In this section, the Cayley graphs associated to the dihedral group of order six and eight with subsets S of order one are constructed. From Definition 1, $D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$, while the dihedral group of order eight, $D_8 = \langle R_0, R_{90}, R_{180}, R_{270}, V, D', H, D \rangle$. By Definition 2, the subsets of D_6 of order one are $\{L_1\}, \{L_2\}$ and $\{L_3\}$, while the subsets of D_8 of order one are $\{V\}, \{D'\}, \{H\}, \{D\}$ and $\{R_{180}\}$. The results of Cayley graphs are presented in two propositions.

Proposition 1 Let D_6 be the dihedral group of order six. Then, the Cayley graph of D_6 with subset *S* of order one, $Cay(D_6, S) = 3K_2$.

Proof Let D_6 be the dihedral group of order six, $D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$, and $Cay(D_6, S)$ be the Cayley graph of D_6 with subsets S of order one, namely $\{L_1\}, \{L_2\}$ and $\{L_3\}$. First, let $S = \{L_1\}$. By Definition 2, the vertex set of the Cayley graph, $V(Cay(D_6, \{L_1\})) = D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Next, to determine the edges of the graph, the Cayley table of D_6 (Table 1) is used. From Definition 2, there is an edge joining the vertices v_1 and v_2 if and only if $v_2 = sv_1$ for $s \in \{L_1\}$, and v_1, v_2 in D_6, or in short $v_2 = L_1 v_1$.

Table 1 The Cayley table of D_6 for subset $\{L_1\}$

	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_0	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_{120}	R_{120}	R_{240}	R_0	L_3	L_1	L_2
R_{240}	R_{240}	R_0	R_{120}	L_2	L_3	L_1
L_1	L_1	L_2	L_3	R_0	R_{120}	R_{240}
L_2	L_2	L_3	L_1	R_{240}	R_0	R_{120}
L_3	L_3	L_1	L_2	R_{120}	R_{240}	R_0

In Table 1, the orange highlighted rows represent v_1, the yellow highlighted rows represent v_2, while the green highlighted cell represents $s \in \{L_1\}$, which show that $v_2 = sv_1$, $s \in \{L_1\}$. Since $L_1 = L_1 R_0$, thus R_0 is adjacent to L_1. Since $L_2 = L_1 R_{120}$, thus R_{120} is adjacent to L_2. Since $L_3 = L_1 R_{240}$, thus R_{240} is adjacent to L_3. Thus, the edge set of the Cayley graph, $E(Cay(D_6, \{L_1\})) = \{\{R_0, L_1\}, \{R_{120}, L_2\}, \{R_{240}, L_3\}\}$. Hence, $Cay(D_6, \{L_1\}) = 3K_2$, which can be drawn as in Figure 1.

R_0 R_{120} R_{240}

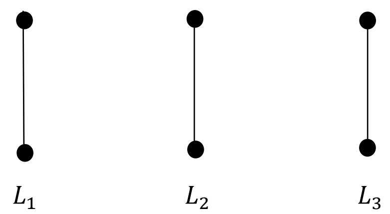

Figure 1 The Cayley graph of D_6 with the subset $\{L_1\}$.

The proof for $Cay(D_6, \{L_2\})$ and $Cay(D_6, \{L_3\})$ is similar to that of $Cay(D_6, \{L_1\})$. Hence, the Cayley graph of D_6 with subset S of order one, $Cay(D_6, S) = 3K_2$. □

Proposition 2 Let D_8 be the dihedral group of order eight. Then, the Cayley graph of D_8 with subset S of order one, $Cay(D_8, S) = 4K_2$.

Proof Let D_8 be the dihedral group of order eight, $D_8 = \langle R_0, R_{90}, R_{180}, R_{270}, V, D', H, D\rangle$, and $Cay(D_8, S)$ be the Cayley graph of D_8 with subsets S of order one, namely $\{V\}, \{D'\}, \{H\}, \{D\}$ and $\{R_{180}\}$. First, let $S = \{V\}$. By Definition 2, the vertex set of the Cayley graph, $V(Cay(D_8, \{V\})) = D_8 = \langle R_0, R_{90}, R_{180}, R_{270}, V, D', H, D\rangle$. Next, to determine the edges of the graph, the Cayley table of D_8 (Table 2) is used. From Definition 2, there is an edge joining the vertices v_1 and v_2 if and only if $v_2 = sv_1$ for $s \in \{V\}$, and v_1, v_2 in D_8, or in short $v_2 = Vv_1$.

Table 2 The Cayley table of D_8 for subset $\{V\}$

	R_0	R_{90}	R_{180}	R_{270}	V	D'	H	D
R_0	R_0	R_{90}	R_{180}	R_{270}	V	D'	H	D
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D	V	D'	H
R_{180}	R_{180}	R_{270}	R_0	R_{90}	H	D	V	D'
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D'	H	D	V
V	V	D'	H	D	R_0	R_{90}	R_{180}	R_{270}
D'	D'	H	D	V	R_{270}	R_0	R_{90}	R_{180}
H	H	D	V	D'	R_{180}	R_{270}	R_0	R_{90}
D	D	V	D'	H	R_{90}	R_{180}	R_{270}	R_0

In Table 2, since $V = VR_0$, thus R_0 is adjacent to V. Since $D' = VR_{90}$, thus R_{90} is adjacent to D'. Since $H = VR_{180}$, thus R_{180} is adjacent to H. Since $D = VR_{270}$, thus R_{270} is adjacent to D. Thus, the edge set of the Cayley graph, $E(Cay(D_8, \{V\})) = \{\{R_0, V\}, \{R_{90}, D'\}, \{R_{180}, H\}, \{R_{270}, D\}\}$. Hence, $Cay(D_8, \{V\}) = 4K_2$, which can be drawn as in Figure 2.

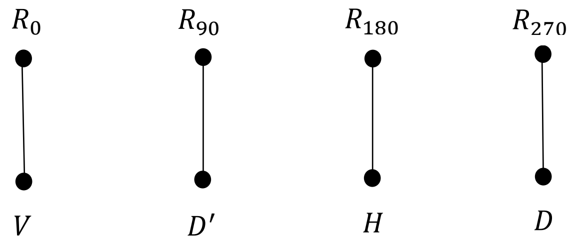

Figure 2 The Cayley graph of D_8 with the subset $\{V\}$.

The proof for $Cay(D_8, \{D'\}), Cay(D_8, \{H\})$ and $Cay(D_8, \{D\})$ is similar to that of $Cay(D_8, \{V\})$. Hence, the Cayley graph of D_8 with subset S of order one, $Cay(D_8, S) = 4K_2$ □

3.2 The Energy of the Cayley Graph Associated to the Dihedral Group of Order Six and Eight with Subsets of Order One

In this section, the energy is computed and presented for the Cayley graph associated to the dihedral group of order six and eight with subsets of order one.

Proposition 3 Let D_6 be the dihedral group of order six. Then, the energy of the Cayley graph of D_6 with subset S of order one, $\varepsilon(Cay(D_6, S)) = 6$.

Proof Let R be a graph, D_6 be the dihedral group of order six and $Cay(D_6, S)$ be the Cayley graph of D_6 with the subset S of order one. First, let $S = \{L_1\}$. By Definition 5 of the adjacency matrix, the rows and columns of $A(R)$ are indexed by $V(R)$, namely $v_1, v_2, v_3, v_4, v_5, v_6$ where $v_1 = R_0, v_2 = R_{120}, v_3 = R_{240}, v_4 = L_1, v_5 = L_2, v_6 = L_3$. Since $Cay(D_6, \{L_1\}) = 3K_2$, hence the corresponding adjacent vertices have the entry 1, otherwise, the entries are 0. Thus, the adjacency matrix of $Cay(D_6, \{L_1\})$ is obtained as follows:

$$A(R) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Then, the characteristic polynomial of $A(R)$,

$$f(A(R), \lambda I) = det(A(R) - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 1 \\ 1 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 1 & 0 & 0 & -\lambda \end{vmatrix} = (\lambda + 1)^3(\lambda - 1)^3.$$

By Definition 6, the eigenvalues are $\lambda_1 = 1$ with multiplicity 3 and $\lambda_2 = -1$ with multiplicity 3. By Definition 7, the energy of $Cay(D_6, \{L_1\})$, $\varepsilon(Cay(D_6, \{L_1\})) = 3|1| + 3|-1| = 6$. The proof for $\varepsilon(Cay(D_6, \{L_2\}))$ and $\varepsilon(Cay(D_6, \{L_3\}))$ is similar as $\varepsilon(Cay(D_6, \{L_1\}))$. Hence, the energy of the Cayley graph of D_6 with subset S of order one, $\varepsilon(Cay(D_6, S)) = 6$. □

Proposition 4 Let D_8 be the dihedral group of order eight. Then, the energy of the Cayley graph of D_8 with subset S of order one, $\varepsilon(Cay(D_8, S)) = 8$.

Proof The method of the proof is similar to the proof in the previous proposition. □

3.3 The Minimum Degree Energy of the Cayley Graph Associated to the Dihedral Group of Order Six and Eight with Subsets of Order One

This subsection presents the results in the form of propositions on the minimum degree energy of the Cayley graph associated to the dihedral group of order six and eight with subsets of order one.

Proposition 5 Let D_6 be the dihedral group of order six. Then, the minimum degree energy of the Cayley graph of D_6 with subset S of order one, $\varepsilon_M(Cay(D_6, S)) = 6$.

Proof Let R be a graph, D_6 be the dihedral group of order six and $Cay(D_6, \{L_1\})$ be the Cayley graph of D_6 with the subset $\{L_1\}$. By Definition 8 of the minimum degree matrix, the rows and columns of $M(R)$ are indexed by $V(R)$, namely $v_1, v_2, v_3, v_4, v_5, v_6$ where $v_1 = R_0, v_2 = R_{120}, v_3 = R_{240}, v_4 = L_1, v_5 = L_2, v_6 = L_3$. The corresponding minimum degree of a vertex has the entry 1, otherwise 0. This is because $Cay(D_6, \{L_1\})$ is the union of three complete graphs of two vertices, $3K_2$, where the minimum degrees of all vertices of $Cay(D_6, \{L_1\})$ are 1. Thus, the minimum degree matrix of $Cay(D_6, \{L_1\})$ is obtained as follows:

$$M(R) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

It can be seen that the minimum degree matrix and the adjacency matrix of the Cayley graph associated to the dihedral group of order six for the subset $\{L_1\}$ are the same. Since the minimum degree matrix and the adjacency matrix of $Cay(D_6, \{L_1\})$ are the same, the proof follows from that of Proposition 3. Thus, the characteristic polynomial of $M(R)$ is $g(M(R), \lambda I) = (\lambda + 1)^3(\lambda - 1)^3$ and the minimum degree eigenvalues of $Cay(D_6, \{L_1\})$ are $\lambda_1 = 1$ with multiplicity 3 and $\lambda_2 = -1$ with multiplicity 3. Therefore, by Definition 9, $\varepsilon_M(Cay(D_6, \{L_1\})) = 3|1| + 3|-1| = 6$. The proof for $\varepsilon_M(Cay(D_6, \{L_2\}))$ and $\varepsilon_M(Cay(D_6, \{L_3\}))$ is similar as $\varepsilon_M(Cay(D_6, \{L_1\}))$. Hence, the minimum degree energy of the Cayley graph of D_6 with subset S of order one, $\varepsilon_M(Cay(D_6, S)) = 6$. □

Proposition 6 Let D_8 be the dihedral group of order eight. Then, the minimum degree energy of the Cayley graph of D_8 with subset S of order one, $\varepsilon_M(Cay(D_8, S)) = 8$.

Proof The proof is similar to the proof in the previous proposition. □

Conclusion

As a conclusion, the results show that the energy and the minimum degree energy of the Cayley graph associated to the dihedral group of order six and eight are the same. Besides, the energies obtained are all even.

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