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The Total Non-Zero Divisor Graph for the Ring of Gaussian Integers Modulo Four

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Abstract
For a commutative ring R, the total non-zero divisor graph of R is defined as a graph in which its vertices are the non-zero elements of R, and two vertices x and y are adjacent if $xy \neq 0$ and $x + y$ is a zero divisor in R. In this paper, the total non-zero divisor graph is constructed for the ring of Gaussian integers modulo four. First, the zero divisors for the ring are found and the pairs of elements that satisfy the two properties are determined. Then, the total non-zero divisor graph is constructed by using its definition. Finally, two properties of the graph, known as the clique number and the chromatic number are obtained. As a result, the total non-zero divisor graph is found to be perfect.
Keywords: Commutative ring; Gaussian integers; total graph; non-zero divisor graph

Introduction

The study of graphs has become a fundamental pillar in mathematics due to its numerous applications across various fields such as in the field of pure and applied mathematics, cryptography, coding theory, and network analysis. The zero-divisor graph associated to commutative ring was firstly introduced by Beck [1] in 1988, in which the vertices are the elements of the ring and two different elements x and y are adjacent if and only if $xy = 0$. Beck's main focus was on the coloring of the graph. By studying Beck's research on the zero-divisor graph of a commutative ring, Anderson and Livingston [2] introduced the simplified version of the graph when the researchers considered the zero divisors of the ring to be the vertices of the graph. The authors in [2] also studied the connectivity and automorphisms of the graph, offering deeper insights into their graph-structural characteristics.

In 2003, Anderson, Levy and Shapiro [3] studied the zero-divisor graph of von Neumann regular rings and Boolean algebras. Estaji and Khashyarmanesh [4] then extended the concept of zero-divisor graph associated to an arbitrary finite bounded lattice. The researchers studied the structure of the graph and found out that if the zero-divisor graph of a lattice is a complete bipartite graph, then the lattice has exactly two atoms. In 2014, Chelvam and Nithya [5] continued the research on the zero-divisor graph associated to a lattice by studying the graph-theoretic properties such as diameter, girth, radius, dominating set and dominating number. The zero-divisor graph of various groups and rings has also been of interest of several researchers, see [6-10].

Meanwhile, the total graph of a commutative ring was introduced by Anderson and Badawi [11] in 2008 as the graph with vertices of the elements of a ring R and there exists an edge between two distinct vertices $x, y \in R$ if $x + y$ is a zero-divisor in R. This graph engages both multiplication and addition ring operations, therefore reflects more structure of the subset of zero-divisors in a given ring. Later in 2019, Duric *et al.* [12] introduced the total zero divisor graph, in which the vertices are the non-zero zero divisors of the ring and two distinct vertices x and y are adjacent if and only if $xy = 0$ and $x + y \in Z(R)$, the set of zero divisors. In [12], they investigated the connectedness and graph-theoretic properties of the total zero-divisor graph associated to commutative rings. Recently, Visweswaran [13] studied the complement of the total zero-divisor graph of a commutative ring with identity.

On the other hand, in 2021, non-zero divisor graph has been defined [14] as a simple graph with its set of vertices consists of all non-zero elements of the ring and there exists an edge between two distinct vertices x, y when $xy \neq 0$. Research on the non-zero divisor graph has been studied extensively on different groups and rings such as the Hamiltonian quaternion and Gaussian integers. Recently in 2023, Zai et al. [15] introduced the total non-zero divisor graph of some commutative rings.

In this research, the concept of the total non-zero divisor graph of a commutative ring, in specific the ring of Gaussian integers modulo four is studied. Two properties of the graph such as the clique number and chromatic number are found. Then, the perfectness of the graph is determined.

Preliminaries

In this section, some definitions and properties related to the main topic are presented.

Definition 1 [16] Zero Divisors

Zero divisors of a ring are the non-zero elements that have product zero when multiplied with each other.

Next, some definitions related to graph theory are presented.

Definition 2 [17] Graph

A finite graph, denoted as $\boldsymbol{\Gamma}$ is an object with two sets, which are the edge set, $\boldsymbol{E}(\boldsymbol{\Gamma})$ and the non-empty vertex set, $\boldsymbol{V}(\boldsymbol{\Gamma})$.

The formal definition of the total non-zero divisor graph is given in the following.

Definition 3 [15] Total Non-Zero Divisor Graph

A total non-zero divisor graph of a ring $\boldsymbol{R}$, denoted by $\boldsymbol{\Gamma_T}(\boldsymbol{R})$ is a simple undirected graph whose vertex set is the set of all non-zero elements of $\boldsymbol{R}$, and two distinct vertices $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent if and only if $\boldsymbol{xy \neq 0}$ and $\boldsymbol{x + y}$ is a zero divisor in $\boldsymbol{R}$.

In this research, the total non-zero divisor graph is constructed for the ring of Gaussian integers modulo four. The ring of Gaussian integer is defined in the following.

Definition 4 [16] Ring of Gaussian Integer

The ring of Gaussian integer, $\mathbb{Z}(\boldsymbol{i}) = \{a + bi \mid a, b \in \mathbb{Z}\}$.

In specific, the ring of Gaussian integer modulo n, $\mathbb{Z}_{\boldsymbol{n}}(\boldsymbol{i}) = \{a + bi \mid a, b \in \mathbb{Z}_{\boldsymbol{n}}\}$.

Then, some properties of the total non-zero divisor graph of the ring of Gaussian integer modulo four, namely the chromatic number and the clique number are determined. The definitions of the clique number and chromatic number are given as follows.

Definition 5 [18] Clique

A clique is a complete subgraph in a graph, $\boldsymbol{\Gamma}$.

Definition 6 [19] Clique Number

A clique number, usually denoted by $\boldsymbol{\omega}(\boldsymbol{\Gamma})$, is the greatest size of a clique in an undirected graph $\boldsymbol{\Gamma}$.

Definition 7 [19] Chromatic Number

The chromatic number, usually denoted by $\boldsymbol{\chi}(\boldsymbol{\Gamma})$, is the minimum amount of colors needed to color the vertices of $\boldsymbol{\Gamma}$ so that no two neighboring vertices share the same color.

Definition 8 [20] Perfect Graph

A perfect graph is a graph Γ for which every induced subgraph of Γ has chromatic number equal to its clique number.

Results and Discussions

In this section, the results on the total non-zero divisor graph constructed for the ring of Gaussian integers modulo four, along with the graph-theoretic properties, are presented.

Proposition 1 The total non-zero divisor graph of the ring of Gaussian integers modulo four, denoted by $\Gamma_T(\mathbb{Z}_4[i])$, is a simple, undirected graph with 15 vertices and 36 edges.

Proof. The ring of Gaussian integers modulo four, $\mathbb{Z}_4[i] = \{0,1,2,3, i, 2i, 3i, 1 + i, 1 + 2i, 1 + 3i, 2 + i, 2 + 2i, 2 + 3i, 3 + i, 3 + 2i, 3 + 3i\}$. Since there are 15 non-zero elements of the ring, by Definition 3, there are 15 vertices of its total non-zero divisor graph of the ring of Gaussian integers modulo four, $\Gamma_T(\mathbb{Z}_4[i])$. To determine the edges of $\Gamma_T(\mathbb{Z}_4[i])$, by Definition 3, x is adjacent to y if $xy \neq 0$ and $x+y$ is a zero divisor in $\mathbb{Z}_4[i]$. First, the set of zero divisors of $\mathbb{Z}_4[i]$ is found as $\{2, 2i, 1+i, 1+3i, 2+2i, 3+i, 3+3i\}$. Next, by going through all multiplication and addition of the elements in the ring, the pair of elements

that satisfy the two conditions are determined. Then, the total number of edges of the total non-zero divisor graph of $\mathbb{Z}_4[i]$ is found to be 36. Hence, the graph can be constructed as in Figure 1.

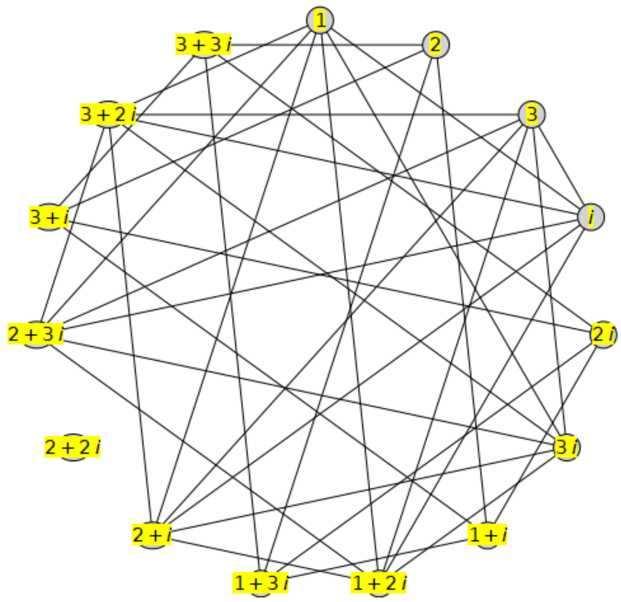

Figure 1. The total non-zero divisor graph $\Gamma_T(\mathbb{Z}_4[i])$

Next, the properties of the graph are determined.

Proposition 2 The clique number and the chromatic number of $\Gamma_T(\mathbb{Z}_4[i])$ are four.

Proof. Using Definition 5, the set of maximum cliques of the total non-zero divisor graph of the ring of Gaussian integers modulo four, $\Gamma_T(\mathbb{Z}_4[i])$ is $\{1, i, 1+2i, 2+i\}$. So, by Definition 6, the clique number of $\Gamma_T(\mathbb{Z}_4[i])$ is four. There are four colors that can be applied on the vertices of $\Gamma_T(\mathbb{Z}_4[i])$ so that no two adjacent vertices share the same color. Therefore, by Definition 7, the chromatic number, $\chi(\Gamma_T(\mathbb{Z}_4[i]))$ is four. Figure 2 shows the graph $\Gamma_T(\mathbb{Z}_4[i])$ with colors.

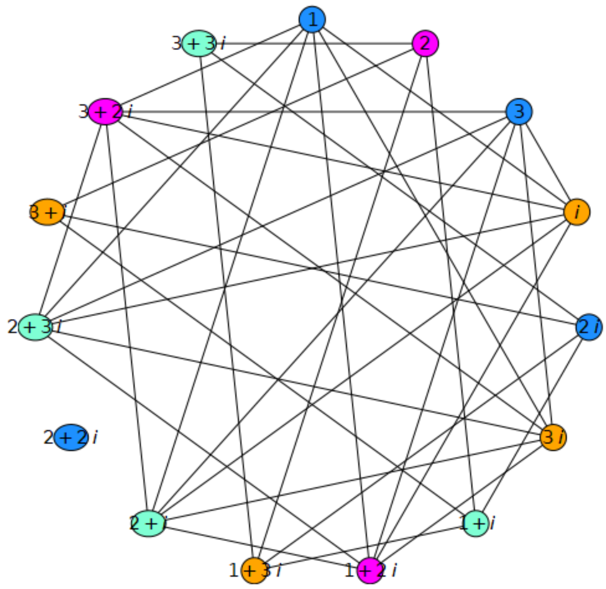

Figure 2. The total non-zero divisor graph $\Gamma_T(\mathbb{Z}_4[i])$ with colors

Based on the two propositions above, the graph is perfect, stated formally in the following proposition.

Proposition 3 The total non-zero divisor graph of the ring of Gaussian integers modulo four, $\Gamma_T(\mathbb{Z}_4[i])$ is perfect.

Proof. Based on Proposition 2, $\chi\big(\Gamma_T(\mathbb{Z}_4[i])\big) = \omega\big(\Gamma_T(\mathbb{Z}_4[i])\big) = 4$. Since the clique number is equal to the chromatic number, by Definition 8, the total non-zero divisor graph of the ring of Gaussian integers modulo four, $\Gamma_T(\mathbb{Z}_4[i])$ is perfect.

Conclusion

In conclusion, the construction of the total non-zero divisor graph of the ring of Gaussian integers modulo four, $\Gamma_T(\mathbb{Z}_4[i])$ has provided valuable insights to its structural properties. The total non-zero divisor graph is found to be a simple, undirected graph with 15 vertices and 36 edges. This graph exhibits interesting characteristic which is having the clique number equal to the chromatic number, hence making it a perfect graph. In future work, researchers can explore additional graph properties, investigating the relationship between total non-zero divisor graphs and other algebraic structures, and potentially extending the analysis to different groups and rings.

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