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Dynamical Behaviors of Prey-Predator Model with Allee Effect and Linear Harvesting

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Abstract
In this paper, we propose a modified prey-predator model with Holling Type II functional response that incorporates with the factors of Allee effect and linear harvesting. The phenomenon of Allee effect occurs among the prey species while the assumption of harvesting the predator species is also taken into account in this research. The modified prey-predator model incorporating with Allee effect and linear harvesting is developed from the classical Lotka-Volterra prey-predator model. A few assumptions are made before we derive the modified prey-predator model. The nondimensionalization method is carried out to reduce the number of parameters involved in the model and ease us for a better interpretation and computation. The stability analysis of each equilibrium point is conducted in this research. The stability of extinction and predator-free equilibria is determined by using the eigenvalue of Jacobian matrix while the stability of the coexistence equilibrium is determined by using Routh-Hurwitz stability criterion. The bifurcation analysis is also addressed. The bifurcation with respect to the harvesting parameter is conducted. Based on the research, Hopf and transcritical bifurcations are detected when the parameter is varied. At the low level of harvesting activities, the Hopf bifurcation occurs, while the transcritical bifurcation occurs at the high level of harvesting activities. The coexistence equilibrium is stable at the intermediate level of harvesting activities. Numerical simulations of the model are performed to demonstrate the dynamical behaviors of the modified prey-predator model. The bifurcation diagram with respect to linear harvesting parameter is obtained by using MATLAB packages to investigate the dynamical behaviours of the model when both parameters are varied. The time series are also obtained to verify the results obtained from the bifurcation analysis.
Keywords: Prey; Predator; Allee Effect; Linear Harvesting; Holling Type II

1. Introduction

In the era of urbanization and modernization, humans emphasize more on the healthy lifestyle, especially the balanced diet. Protein is one of the essential nutrition required by the human body to boost the growth rate during the teenage year and aid in tissue repair. Humans harvest a variety of animals in order to ensure the protein supply. The statistics show that a significant increase in fish consumption from 9 kg per capita in the 1960s to 20.5 kg per capita in 2018 [1]. The data verifies the statement that the humans have a great dependency on fishery harvesting activities to sustain the protein resource. Harvesting activities are conducted to catch various types of animals and increase the yield of protein resources. Around 10% of the global population depends on small-scale fisheries for them to meet the daily requirements of protein [2]. On the other hand, excessive harvesting activities cause over exploitation of protein resources that might threaten the supply chains of the protein resource and food security. This situation threatens the balance of the ecosystem as it can cause the extinction and disappearance of certain species and influence the dynamics of the natural environment. Therefore, it is vividly obvious that the frequency of harvesting activities should be monitored at an appropriate level.

The excessive harvesting activities cause a biological phenomenon called the Allee effect which is proposed by W.C. Allee in the year 1930s [3]. Allee proposed that the intraspecific cooperation might lead to inverse density dependence, which is also known as Allee effect. Allee observed that many animal and plant species experienced a reduction in their per capita growth rate when their populations reached small sizes or low densities [4]. This implies that the over harvesting declines the animal population and decrease the chance for the animals to find a mate and reproduce their offsprings [5]. The low reproduction rate makes the situation of low population. According to [5], Allee effects can be classified into two types of Allee effect which are component Allee effect and demographic Allee effect. Component Allee effect is defined as a positive correlation between any measurable component of individual fitness and the size of the population while demographic Allee effect is defined as a positive correlation between the overall individual fitness and the size of the population [6]. It is very clear that the Allee effect might reduce the population of certain species and lead to extinction which is a great loss to mankind.

Therefore, during the formulation of the prey-predator model, we should take the harvesting activities and Allee effect into account in order to research the dynamical behaviors when the parameters values of the system are varied. This is very essential for us to determine the appropriate frequency level of harvesting activities under the influence of the Allee effect so that the undesired scenario which is the extinction of either one or both species does not occur in the real life. Many researchers are now considering including the Allee effect and harvesting activities in the prey predator model. Experimental evidence of the Allee effect in the natural environment is provided by AlSharawi et al.. [7] Allee effect can be categorized into strong Allee effect and weak Allee effect [8]. The model developed by Wei and Chen et al. considers the strong Allee effect occurring on the prey species [9]. The model developed by Eskandari et al. redefines the function of Allee effect in terms of individual searching efficiency [10]. Both models describe the single Allee effect implemented on the prey species, however the difference between fertile and non-fertile is not described [11].

Harvesting is one of the simple approaches to obtain food resources in the modern era. Harvesting activities can be categorized into three types. The first type of harvesting is constant harvesting, which implies a certain number of species that can be harvested at a period of time. The second linear is linear harvesting, as known as the proportional harvesting. We introduce the variables called harvesting effort and catchability coefficient. This type of harvesting activity implies the number of species harvested is proportional to time. And the last type of harvesting is nonlinear harvesting which is the Holling type II harvesting because it has a similar parameter as the Holling type II functional response. Other than the harvesting effort and catchability coefficient, the parameter introduced in the nonlinear harvesting is the handling time for the harvested species, which implies the time taken to handle the victim [12].

The main objective of this research paper is to study the dynamical behaviors of a modified prey-predator model by incorporating the factor of the Allee effect and linear harvesting. The Allee effect is implemented on the prey species and the predator species is subject to a certain rate of harvesting. We want to study the effect of the Allee effect to enhance the dynamical behavior of the modified prey-predator model.

Model Formulation

We derive the modified prey-predator model from the classical prey-predator model, which is as known as Lotka-Volterra model [13] that takes the following form:

$$\begin{aligned} \frac{dX}{dt} &= aX - bXY, \\ \frac{dY}{dt} &= cXY - dY, \end{aligned} \tag{1}$$

where X represents the prey density, Y represents the predator density and t represents the time. The prey grows at the rate of a when the predator is absent, while the predator decline at the rate of d when the prey is absent. The parameters b and c represent the rate of interaction between the prey and predator when both species exist. However, this seems illogical for the prey to grow exponentially as the resource in the environment is limited and to resolve this situation, the logistical growth model is implemented in the prey-predator model and the variable of environmental carrying capacity is introduced [14]. In order to increase the complexity of the modified prey-predator model, the Holling type II functional response which includes the handling time on the prey species by the predator is implemented in the model [15]. The modified prey-predator model becomes:

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right)-\frac{eXY}{1+bx}, \qquad (2)$$
$$\frac{dY}{dt} = \frac{CeXY}{1+bx}-dY,$$

where K is the environmental carrying capacity, e is the attacking rate of the predator and b is the handling time of predator. The growth rate of prey population is assumed to grow logistically with the rate of r when the predator is absent while the predator population reduces at the rate of d when the prey is absent. Next, we add the terms related to the Allee effect on the prey and the harvesting parameter on the predator [10, 12].

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right)\left(\frac{X}{X+A}\right)-\frac{eXY}{1+bx}, \qquad (3)$$
$$\frac{dY}{dt} = \frac{CeXY}{1+bx}-dY-qEY,$$

where A is individual searching efficiency, C is the conversion efficiency to grow a predator from the prey, q is the catchability coefficient of predator populations, and E is the harvesting effort on the predator. The catch rate function qEY follows the catch-per-unit-effort rules. Allee effect is only implemented on the prey species in this model as the predator consumes the prey and cause a decline in the prey population that may lead to the low population in prey and trigger the Allee effect. All parameters are assumed to be positive. Therefore, the Allee effect applied on the prey species is the strong single Allee effect.

Non-dimensional Model

To reduce the number of parameters involved in the modified prey-predator model for a better interpretation, non dimensionalisation is conducted by using a set of scaled variables of

$$x = \frac{CeX}{d}, y = \frac{eY}{d}, \tau = dt$$

The dimensional system (3) becomes

$$\frac{dx}{dt} = \alpha x(1-\beta x)\left(\frac{\rho x}{\rho x+\varepsilon}\right)-\frac{xy}{1+\mu x}, \qquad (4a)$$

$$\frac{dy}{dt} = -y+\frac{xy}{1+\mu x}-\delta y, \qquad (4b)$$

where

$$\alpha = \frac{r}{d}, \beta = \frac{d}{CeK}, \rho = \frac{d}{Ce}, \mu = \frac{bd}{Ce}, \delta = \frac{qE}{d}, \varepsilon = A.$$

In the system (4),

α : ratio of the growth rate of x to the death rate of y

β : ratio of the death rate of y to the product of the conversion efficiency of y, attacking rate of y and the environmental carrying capacity of x.

ρ : ratio of the death rate of y to the product of the conversion efficiency of y and attacking rate of y

μ : ratio of the product of the handling time of y and death rate of y to the product of the conversion efficiency of y, attacking rate of y

δ : ratio of the product of catchability coefficient and the harvesting effort of y to the death rate of y

ε : individual searching efficiency

Steady State, Equilibria and Stability Analysis

To obtain the equilibrium points of system (4), we can solve the system by letting $\frac{dx}{d\tau} = \frac{dy}{d\tau} = 0$. There are three possible steady state of P_i in the form of (x,y), where *i* represents the number of steady states occur in the system and two of them are the trivial steady states:

$$P_1=(0,0), P_2=\left(\frac{1}{\beta},0\right), P_3=\left(-\frac{\delta+1}{\mu(\delta+1)-1}, \frac{\alpha\rho(\delta+1)\left[(\mu+\beta)(\delta+1)-1\right]}{\left[\varepsilon\left(\mu(\delta+1)-1\right)-\rho(\delta+1)\right]\left(\mu(\delta+1)-1\right)^2}\right)$$

Next, we can inspect the stability of equilibrium point by using the Jacobian matrix, where the system can be generalized to

$$J=\begin{bmatrix} \frac{2\alpha\rho x(1-\beta x)}{\rho x+\varepsilon}-\frac{\alpha\beta\rho x^2}{\rho x+\varepsilon}-\frac{\alpha\rho^2x^2(1-\beta x)}{(\rho x+\varepsilon)^2}-\frac{y}{\mu x+1}+\frac{\mu xy}{(\mu x+1)^2} & -\frac{x}{\mu x+1} \\ \frac{y}{\mu x+1}-\frac{\mu xy}{(\mu x+1)^2} & \frac{x}{\mu x+1}-1-\delta \end{bmatrix} \quad (5)$$

The equilibrium P_1 represents the extinction of both prey and predator species. By substituting the value of P_1 into (5), we obtain the following Jacobian matrix of $J_{P_1}=\begin{bmatrix} 0 & 0 \\ 0 & -1-\delta \end{bmatrix}$, which gives the following characteristic equation of

$$\lambda^2+(\delta+1)\lambda=0. \quad (6)$$

where λ is the eigenvalue. Equilibrium P_1 gives a set of eigenvalues of $\{0,-1-\delta\}$. By computing the determinant of the characteristic equation (5), since we know that $(1+\delta)$ is always positive, therefore we can conclude that P_1 is a stable node.

The equilibrium P_2 represents the extinction of the predator species, which is known as the predator-free equilibrium. By substituting the value of P_2 into (5), we obtain the following Jacobian matrix of $J_{P_2}=\begin{bmatrix} -\frac{\alpha\rho}{\rho+\beta\varepsilon} & -\frac{1}{\mu+\beta} \\ 0 & \frac{1}{\mu+\beta}-1-\delta \end{bmatrix}$ which gives the following characteristic equation of

$$\lambda^2+\frac{\left[\alpha\rho(\mu+\beta)+(\varepsilon\beta+\rho)(\beta\delta+\delta\mu+\beta+\mu-1)\right]}{(\varepsilon\beta+\rho)(\mu+\beta)}\lambda+\frac{\left[\alpha\rho(\beta\delta+\delta\mu+\beta+\mu-1)\right]}{(\varepsilon\beta+\rho)(\mu+\beta)}=0. \quad (7)$$

Equilibrium P_2 gives a set of eigenvalues of $\left\{-\frac{(\mu+\beta)(\delta+1)-1}{\mu+\beta}, -\frac{\alpha\rho}{\varepsilon\beta+\rho}\right\}$. Since all parameter values are assumed to be positive, $-\frac{\alpha\rho}{\varepsilon\beta+\rho}$ is always negative. Hence, we can determine the stability criterion of P_2 based on the value of $-\frac{(\mu+\beta)(\delta+1)-1}{\mu+\beta}$ and here are the appropriate conditions:

$$(\mu+\beta)(\delta+1)>1, \quad \text{[i]}$$

$$(\mu+\beta)(\delta+1)<1. \quad \text{[ii]}$$

If condition [i] holds, then the equilibrium P_2 is guaranteed to be a stable node. Else, if condition [ii] holds, then the equilibrium P_2 is guaranteed to be an unstable node.

The equilibrium P_3 which is the non-trivial steady state represents the coexistence of both prey and predator species. To ensure the feasibility and positivity of the steady state, the following conditions should be obeyed.

$$\mu(\delta+1)-1<0, \quad \text{[iii]}$$

$$(\mu+\beta)(\delta+1)-1<0, \quad \text{[iv]}$$

$$(\varepsilon\mu-\rho)(\delta+1)-\varepsilon<0. \qquad [v]$$

It is tedious to use the Jacobian matrix to examine the stability analysis of equilibrium P_3 Hence, we use Routh-Hurwitz stability criterion to determine the stability condition for this equilibrium point. For a second order system, the sufficient condition of a system holds the same as the necessary condition of the system. To guarantee the stability of the steady state, all the coefficients involved in the characteristic equation should be strictly positive, which indicates that

$$\lambda^2+a_1\lambda+b_1=0. \qquad (8)$$

where

$$a_1=\frac{\varepsilon\mu^3(\delta+1)^3}{\left[\mu(\delta+1)-1\right]\left[(\varepsilon\mu-\rho)(\delta+1)-\varepsilon\right]^2}-\frac{\mu(\delta+1)\left[\mu(\delta+1)\{(\rho-\beta\varepsilon)(\delta+1)+\varepsilon\}+\beta\rho(\delta+1)^2-\{(\rho-\beta\varepsilon)(\delta+1)+\varepsilon\}\right]}{\left[\mu(\delta+1)-1\right]\left[(\varepsilon\mu-\rho)(\delta+1)-\varepsilon\right]^2}-\frac{\beta(\delta+1)\left[2\varepsilon+\rho(\delta+1)\right]+\varepsilon}{\left[\mu(\delta+1)-1\right]\left[(\varepsilon\mu-\rho)(\delta+1)-\varepsilon\right]^2},$$

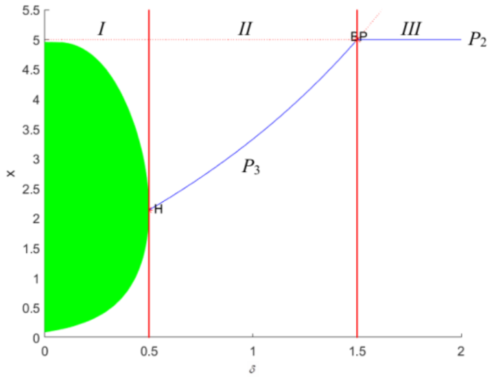
$$b_1=\frac{\rho\alpha(\delta+1)^2\left[(\mu+\beta)(\delta+1)-1\right]}{(\varepsilon\mu-\rho)(\delta+1)-\varepsilon},$$

To ensure the stability of the non-trivial steady state, the following condition should be obeyed:

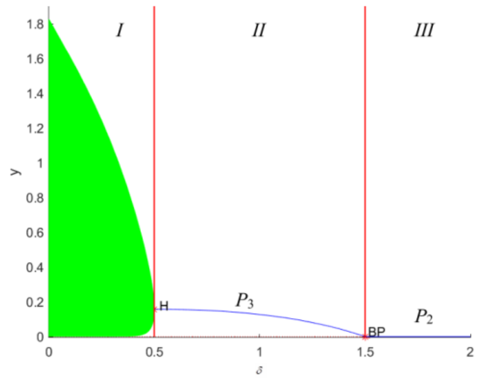
$$\left[\mu(\delta+1)^2(\varepsilon\mu-\rho)-\varepsilon\right]\left[(\mu+\beta)(\delta+1)-1\right]<\beta(\delta+1)\left[\varepsilon-(\varepsilon\mu-\rho)(\delta+1)\right]. \qquad [iv]$$

Bifurcation Results and Analysis

To examine the dynamical behaviors of system (4), the parameter variation technique is used with the aid of MATCONT package in the MATLAB [16]. For simplicity, we set the parameters $\alpha=0.35$, $\beta=0.2$, $\rho=0.4$, $\mu=0.2$ and $\varepsilon=0.7$ In this section, we set δ as the free parameter in the equation (4b) to examine the effects of the harvesting activities on the dynamics of system (4). Figures 1 (a) and (b) show the bifurcation diagrams with respect to the linear harvesting parameter, δ. In both diagrams. For illustrating purpose, the blue solid lines denote stable steady states, whereas the red dashed line denotes unstable steady states. The green solid line denotes the limit cycle of the prey and predator population.



(a)



(b)

Figure 1 Bifurcation curve with respect to the predation-harvesting parameter δ with fixed parameters $\alpha=0.35$, $\beta=0.2$, $\rho=0.4$, $\mu=0.2$ and $\varepsilon=0.7$ for **(a)** prey, x and **(b)** predator, y respectively.

From Figures 1 (a) and 1 (b), there is a Hopf bifurcation (δ = 0.4996) where the population of both prey and predator species oscillates. This is due to the real part of eigenvalues of the characteristic equation (8) approximates 0 when the value of δ < 0.4996. Plus, we can observe a transcritical bifurcation (δ = 1.5) where the stability of two steady states is interchanged. This is because when the value of δ is greater than 1.5, this causes the condition [i] to be satisfied and contract the condition [vi]. Therefore, their stability is interchanged.

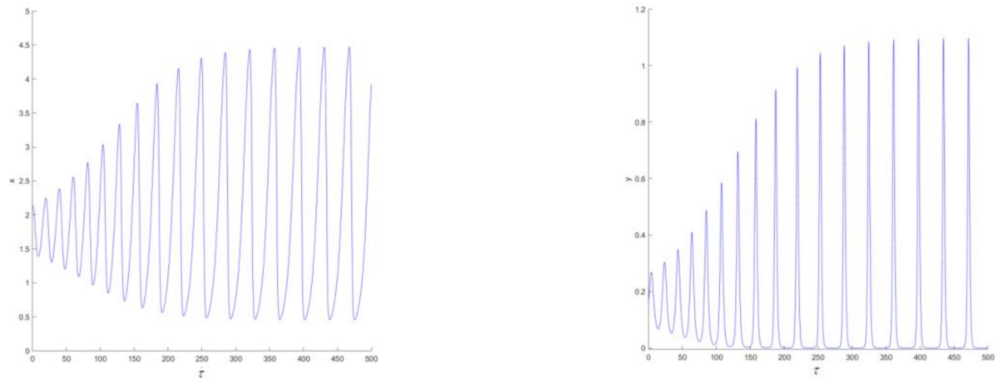
The Hopf and transcritical bifurcation points divide the positive quadrant of the graph into three different regions: I, II and III. Region I represents a low level of harvesting activities, while region II represents an intermediate level of harvesting activities and region III represents a high level of harvesting activities.

Based on Figures 1(a) and (b), the population of prey and predator species both oscillates when a low level of linear harvesting on the predator δ is carried out. When the predation-harvesting activities are conducted at a low rate, the predator species grows due to sufficient food resources as the prey species grows logistically restricted by environmental carrying capacity and Allee effect. At the same time, the prey species decreases when there is a high saturation in prey density. After that, the population of predators decreases due to the decline in food resources, hence, and the population of prey gradually increases again due to decrease in the predation activities. The situation keeps continuing and causing fluctuations in the prey and predator species.

When the harvesting activities are conducted at an intermediate rate which is the value of δ falls in region II, this indicates that the number of predators harvested increases and gives more spaces for the prey species to grow. This predator population gradually increases after the prey population increases as there are sufficient food resources for the predator population to grow. Next, the prey population decreases as they have been consumed by the predator. At this stage, both prey and predator species can coexist in the environment. From the bio-ecological aspect, harvesting activities at this rate are encouraged and ideal as it ensures the survival of both prey and predator and sustains the balance of ecosystems at the same time.

When the harvesting activities are conducted at a high rate which implies the harvesting parameter δ reaches the value of greater than 1.5, the predator-free equilibrium of system (4) converges, which indicates that the equilibrium point P_2 is asymptotically stable while at the same time, the coexistence equilibrium becomes unstable. This is because the predator species is experiencing excessive harvesting, causing it to become extinct from the environment. The prey species continues to grow without any predation. The prey population stops growing when it reaches the high saturation that is determined by the environmental capacity and Allee effect.

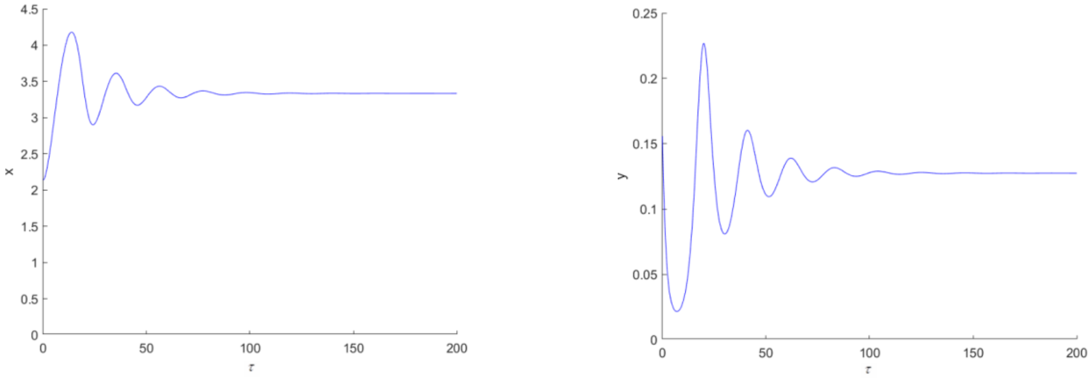
To investigate the underlying trends and dynamics of the prey and predator populations in system (4) over time, the time series plot is obtained by using the MATCONT package in the MATLAB, using the same values for the fixed parameters and by setting δ = 0.3, which is the value in Region I, δ = 1, which is the value in Region II and δ = 2, which is the value in Region III. In Figure 2(a), the number of preys initially decreases due to the predation activity and at the same time, the number of predators increases as shown in Figure 2(b). Due to the harvesting activities, the number of predators decreases and therefore the prey population grows. The process keeps continuing and causes an oscillating pattern in both populations. The change in the amplitude shows that the gap between the prey and predator populations becomes wider and the amplitude is restricted by the environment carrying capacity. However, this scenario is not ideal as if there happens any situation in this environment, for instance, the disease happens on either one species, the population of either one or both species might lead to extinction. Therefore, it is unpredictable and not ideal to harvest the predator at the low level.



(a) **(b)**

Figure 2 Time series plots of system (4) with fixed parameters $\alpha = 0.35$, $\beta = 0.2$, $\rho = 0.4$, $\mu = 0.2$ and $\varepsilon = 0.7$ and the initial condition $(x_0, y_0) = (2.14, 0.157)$, with $\delta = 0.3$ (Region I) for **(a)** prey, *x* and **(b)** predator, *y* respectively

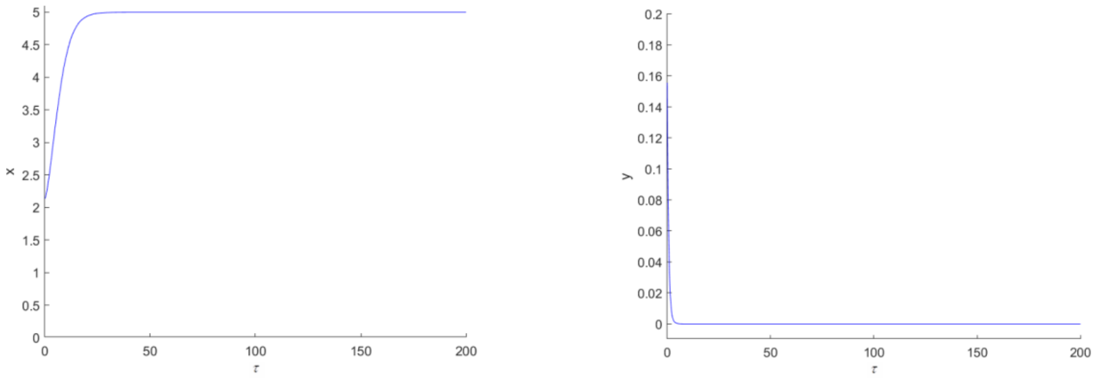
Figures 3(a) and 3(b) describe the phenomena and behaviors of both prey and predator species when the predator harvesting activities are conducted at an intermediate level, which means the value of *δ*. As we can observe from the figures, both prey and predator species coexist at *δ* = 1. The harvesting in this instance does not influence the existence and stability of the coexistence equilibrium point. In the first place, the prey population decreases as the prey is consumed by the predator. After some time, the population of predators decreases due to the harvesting activities, and thus the predation rate on the prey population is reduced. Afterwards, the prey population again increases. The alternation between the prey and predator population progresses until it reaches a stable level. Even though the predator population is being harvested, the high number of prey populations, ensures the existence of the predator species because the predator population only requires a short period of time to hunt the prey and thus, the predator population remains to increase. Both populations can survive at the intermediate level of harvesting activities for a long period of time.



(a) **(b)**

Figure 3 Time series plots of system (4) with fixed parameters $\alpha = 0.35$, $\beta = 0.2$, $\rho = 0.4$, $\mu = 0.2$ and $\varepsilon = 0.7$ and the initial condition $(x_0, y_0) = (2.14, 0.157)$, with $\delta = 1$ (Region II) for **(a)** prey, *x* and **(b)** predator, *y* respectively

When the harvesting parameter *δ* > 1.5, the predator population goes extinct, as shown in Figure 4. This is because, in this state, the predator population has been over exploited, and the prey population keeps growing as the predator population decreases. However, the prey population is restricted to the environmental carrying capacity and Allee effect.



(a) **(b)**

Figure 4 Time series plots of system (4) with fixed parameters $\alpha = 0.35$, $\beta = 0.2$, $\rho = 0.4$, $\mu = 0.2$ and $\varepsilon = 0.7$ and the initial condition $(x_0, y_0) = (2.14, 0.157)$, with $\delta = 2$ (Region III) for **(a)** prey, *x* and **(b)** predator, *y* respectively

Conclusion

This research analyzed the prey-predator population model with Allee effect and linear harvesting for the predator population. The stability properties of the proposed prey-predator model are evaluated at predator-free equilibrium by inspecting the eigenvalues of the characteristic polynomial and coexistence equilibrium by using the Routh-Hurwitz criterion.

The impact of linear harvesting on the prey-predator model is observed with different values of the harvesting parameter, δ, the portrays its influence on both prey and predator populations. Furthermore, the numerical simulation has run on the proposed prey-predator model by using the ODE45 function in MATLAB to illustrate the bifurcation diagram and the time series plot that helps us to visualize the dynamical behaviors of prey and predator populations.

Referring to the steady state diagrams, there is a phenomenon called Hopf bifurcation that causes the oscillation in the prey and predator populations at the low level of the harvesting activities. The phenomenon is not recommendable as it might lead to the extinction of either one or both population species as if there exists an external factor happening to the environment. Next, at the intermediate level of harvesting activity, both species may coexist. Plus, from the time series plots, we can observe that after some certain time, the population of both prey and predator species reach a steady number and remain unchanged after a long period of time. A high level of harvesting should be prohibited as it leads to the extinction of the predator population.

Based on the numerical simulation analysis, the dynamical behavior of the prey and predator population with respect to some critically important parameters has been significantly visualized and illustrated. It is significantly important to highlight how the linear harvesting parameter δ influences the dynamics of the proposed prey-predator model, which can cause the system to emerge in Hopf and transcritical bifurcations. This research also demonstrates how various control parameters can explicitly or implicitly affect the complexity of the prey-predator system.

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