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Some Properties of the Zero Divisor Graph of Integers Modulo Ring

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Abstract
The zero divisor graph of a commutative ring R is a graph in which the set of vertices consists of all zero divisors in R and two vertices are adjacent if their product is zero. In this paper, the zero divisor graph of integers modulo ring with composite order is found for the case $\mathbb{Z}_{p^k q^2}$ where k is a natural number, p and q are prime numbers with $p \neq q$. The edges and vertices of the zero divisor graph for the ring are determined by producing the new propositions and theorem.
Keywords: Zero divisor graph; commutative ring; graph theory; ring theory

Introduction
In graph theory, a graph is a set of vertices (also called nodes or points) connected by edges (also called arcs, links or lines), which can be used to model relationships and structures in various fields such as computer science, biology, and social networks. In 1988, Beck [1] introduced the zero-divisor graph of commutative rings but concerned with colouring of rings. The initial definition was introduced by Anderson and Livingston in [2], which presented numerous fundamental findings regarding to the zero divisor graph of a ring.

Anderson and Mulay [3] investigated the diameter and girth for the zero-divisor graphs of polynomial rings, power series rings, and idealizations. In [4], Anderson and Badawi studied the zero divisor graph of a ring R such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Next, all such zero divisor graphs with 14 or fewer vertices have been determined by Anderson and Weber in [5]. Furthermore, Anderson and McClurkin [6] considered generalizations of $\Gamma(R)$ by modifying the vertices or adjacency relations of $\Gamma(R)$. Recently, Abbasnia et al. [7] introduced and investigated some properties of the new graph for a non-empty subset S of $Z(R)$, denoted by $\Gamma(R, S)$.

This manuscript concentrates on the properties of the zero divisor graph for the integers modulo ring with composite order for the case $\mathbb{Z}_{p^k q^2}$ where k is a natural number, p and q are prime numbers with $p \neq q$.

Preliminaries
In this section, some definitions and concepts in ring theory and graph theory are needed for the discussion are given before moving on to the main findings of this research.

Definition 1 [8] Ring
A ring R is defined as a nonempty set with two compositions $+, \cdot : R \times R \to R$ with the properties:
i) $\langle R, + \rangle$ is an abelian group (with the zero element 0);
ii) $\langle R, \cdot \rangle$ is a semigroup;

iii) for all $a, b, c \in R$ the distributive laws are valid:
$$(a+b)c = ac + bc, a(b+c) = ab + ac.$$

Definition 2 [9] Commutative Ring
A ring R is commutative if $ab = ba$ for all $a, b \in R$.

Definition 1 [10] Zero Divisors of a Ring
If a and b are two nonzero elements of a ring R such that $ab = 0$ then a and b are the zero divisors of R.

Definition 2 [2] Zero Divisor Graph
Let R be a commutative ring with non-zero identity and let $Z(R)$ be the set of all zero divisors of R. The zero divisor graph of R, denoted by $\Gamma(R)$, is an undirected graph whose vertices are elements of $Z\ (R)$ with two distinct vertices a and b are adjacent if and only if $ab = 0$.

Results and discussion

In this section, some results related to the zero divisor graph of integers modulo ring $\mathbb{Z}_{p^k q^2}$, denoted as $\Gamma(\mathbb{Z}_{p^k q^2})$ where $k \in \mathbb{N}$, and p and q are primes with $p \neq q$, are discussed. The following subsections present the zero divisors in the integers modulo ring $\mathbb{Z}_{p^k q^2}$, the zero divisor graph for the integers modulo ring $\mathbb{Z}_{p^k q^2}$ and the calculator application programming code.

Zero Divisors in the Integers Modulo Ring $\mathbb{Z}_{p^k q^2}$

The list of the set of zero divisors in the integers modulo ring $\mathbb{Z}_{p^k q^2}$ is presented in the following proposition.

Proposition 1
The set of all zero divisors in the integers modulo ring $\mathbb{Z}_{p^k q^2}$, is given by
$$\mathrm{Z}(\mathbb{Z}_{p^k q^2}) = \{p, 2p, 3p, 4p, \dots, p(p^{k-1}q^2 - 1)\} \cup \{q, 2q, 3q, 4q, \dots q(p^k q - 1)\}.$$

Proof. Given that $a \in \mathbb{Z}_{p^k q^2}$ then $\mathrm{Z}(\mathbb{Z}_{p^k q^2})$ is presented in the following:

a) Suppose $a \in \mathbb{Z}_{p^k q^2}$ with $\gcd(a, p) > 1$ and $A_1 = \{p, 2p, 3p, 4p, \dots, p(p^{k-1}q^2 - 1)\}$ is $\mathrm{Z}(\mathbb{Z}_{p^k q^2})$ with the cardinality $(p^{k-1}q^2 - 1)$. Furthermore, for $\gcd(a, q) > 1$ and $A_2 = \{q, 2q, 3q, 4q, \dots q, (p^k q - 1)\}$ is $\mathrm{Z}(\mathbb{Z}_{p^k q^2})$ with the cardinality $(p^k q - 1)$.
b) Suppose $a \in \mathbb{Z}_{p^k q^2}$ with $\gcd(a, pq^k) = pq$ and $A_1 \cap A_2$ is $\mathrm{Z}(\mathbb{Z}_{p^k q^2})$. Then, $A_1 \cap A_2 = \{pq, 2pq, 3pq, 4pq, \dots, pq(p^{k-1}q - 1)\}$ with the cardinality $(p^{k-1}q - 1)$.

The set of all zero divisors in the integers modulo ring $\mathbb{Z}_{p^k q^2}$ are as follows:
$$Z(\mathbb{Z}_{p^k q^2}) = A_1 \cup A_2 = \{p, 2p, 3p, 4p, \dots, p(p^{k-1}q^2 - 1)\} \cup \{q, 2q, 3q, 4q, \dots q, (p^k q - 1)\}.$$
□

Proposition 2
The number of zero divisors in the integers modulo ring $\mathbb{Z}_{p^k q^2}$ is given by $|Z(\mathbb{Z}_{p^k q^2})| = q(p^{k-1}(q-1) + p^k) - 1$.

Proof. By using the inclusion-exclusion principle, $|Z(\mathbb{Z}_{p^k q^2})| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.
Then, by using Proposition 1 with their cardinalities, $|Z(\mathbb{Z}_{p^k q^2})| = (p^{k-1}q^2 - 1) + (p^k q - 1) - (p^{k-1}q - 1) = q(p^{k-1}(q-1) + p^k) - 1$. □

Zero Divisor Graph for the Integers Modulo Ring $\mathbb{Z}_{p^k q^2}$

This subsection describes the zero divisor graph for the integers modulo ring $\mathbb{Z}_{p^k q^2}$. Following that, the degree of a vertex a in $\Gamma(\mathbb{Z}_{p^k q^2})$ is separated into five cases as given in Propositions 3, 4, 5, 6, and 7.

Proposition 3
Let $a \in \mathrm{Z}(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i$ for $i = 1,2,3, \dots, k$. Then, $\deg(a) = p^i - 1$.

Proof. Given $\deg(a) = p^i - 1$ with gcd$(a, p^k q^2) = p^i$ for $i = 1,2,3, \dots, k$. Let $a \in \mathrm{Z}(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i$, and let $b \in \mathrm{Z}(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j q^2$ or gcd$(b, p^k q^2) = q^2$ for $i \neq j$ where a and b are adjacent if and only if $i + j \geq k$. Since gcd$(b, p^k q^2) = p^j q^2$ or gcd$(b, p^k q^2) = q^2$ where $j \geq k - i$, so $b \in p^{k-i} q \mathbb{Z}_{p^k q^2}$ and $|p^{k-i} q^2 \mathbb{Z}_{p^k q^2}| = |p^{k-i} q^2 \{0,1,2,3, \dots, p^k q^2 - 1\}| = \frac{p^k q^2}{p^{k-i} q^2} - 1$. Thus, since $\{p^{k-i} q^2 \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$, so $\deg(a) = p^i - 1$. □

Proposition 4 Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = q$. Then, $\deg(a) = q - 1$.

Proof. Given $\deg(a) = q - 1$ with gcd$(a, p^k q^2) = q$. Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = q$, and let $b \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j$ where a and b are adjacent if and only if $j = k$. Since gcd$(b, p^k q^2) = p^j$ where $j = k$, so $b \in p^k q \mathbb{Z}_{p^k q^2}$ and $|p^k q \mathbb{Z}_{p^k q^2}| = |p^k q \{0,1,2,3, \dots, p^k q^2 - 1\}| = \frac{p^k q^2}{p^k q} - 1$. Thus, since $\{p^k q \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$, $\deg(a) = q - 1$. □

Proposition 5 Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = q^2$. Then, $\deg(a) = q^2 - 1$.

Proof. Given $deg(a) = q^2 - 1$ with gcd$(a, p^k q^2) = q^2$. Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = q^2$, and let $b \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j$ and gcd$(b, p^k q^2) = p^j q$ where a and b are adjacent if and only if $j = k$. Since gcd$(b, p^k q^2) = p^j$ and gcd$(b, p^k q^2) = p^j q$ where $j = k$, so $b \in p^k \mathbb{Z}_{p^k q^2}$ and $|p^k \mathbb{Z}_{p^k q^2}| = |p^k \{0,1,2,3, \dots, p^k q^2 - 1\}| = \frac{p^k q^2}{p^k} - 1$. Thus, since $\{p^k \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$, $\deg(a) = q^2 - 1$. □

Proposition 6 Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i q$. Then,

$$\deg(a) = \begin{cases} p^i q - 1, & \text{for } i \leq \left\lfloor \frac{k-1}{2} \right\rfloor, \\ p^i q - 2, & \text{for } i > \left\lfloor \frac{k-1}{2} \right\rfloor. \end{cases}$$

Proof. First, assume $i \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i q$ and let $b \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j q$ and gcd$(b, p^k q^2) = p^j q^2$ where a and b are adjacent if and only if $i + j \geq k$. Since gcd$(b, p^k q^2) = p^j q$ and gcd$(b, p^k q) = p^j q^2$ where $j \geq k - i$, so $b \in p^{k-i} q \mathbb{Z}_{p^k q^2}$ and $|p^{k-i} q Z_{p^k q^2}| = |p^{k-i} q \{0,1,2,3, \dots, p^k q^2 - 1\}| = \frac{p^k q^2}{p^{k-i} q} - 1$. Thus, since $\{p^{k-i} q \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$, $\deg(a) = p^i q - 1$.

Next, assume $i > \left\lfloor \frac{k-1}{2} \right\rfloor$. Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i q$ and let $b \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j q$, gcd$(b, p^k q^2) = p^j q^2$, gcd$(b, p^k q^2) = q$ and gcd$(b, p^k q^2) = q^2$ where a and b are adjacent if and only if $i + j \geq k$. Since gcd$(b, p^k q^2) = p^j q$, gcd$(b, p^k q^2) = p^j q^2$, gcd$(b, p^k q^2) = q$ and gcd$(b, p^k q^2) = q^2$ where $j \geq k - i$, so $b \in p^{k-i} q \mathbb{Z}_{p^k q^2}$ and $\left|p^{k-i} q \mathbb{Z}_{p^k q^2}\right| = \left|p^{k-i} q\{0,1,2,3, \ldots, p^k q^2 - 1\}\right| = \frac{p^k q^2}{p^{k-i} q} - 2$. Thus, since $\{p^{k-i} q \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$ and gcd$(p^i q, p^k q^2) = p^i q$ for $i > \left\lfloor\frac{k-1}{2}\right\rfloor$, so $\deg(a) = p^i q - 2$.

□

Proposition 7 Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i q$. Then,

$$\deg(a) = \begin{cases} p^i q^2 - 1, & \text{for } \; i \leq \left\lfloor\frac{k-1}{2}\right\rfloor, \\ p^i q^2 - 2, & \text{for } \; i > \left\lfloor\frac{k-1}{2}\right\rfloor. \end{cases}$$

Proof. First, assume $i \leq \left\lfloor\frac{k-1}{2}\right\rfloor$. Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i q^2$ and let $b \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j q$, gcd$(b, p^k q^2) = p^j q^2$ and gcd$(b, p^k q^2) = p^j$ where a and b are adjacent if and only if $i + j \geq k$. Since gcd$(b, p^k q^2) = p^j q$, gcd$(b, p^k q^2) = p^j q^2$ and gcd$(b, p^k q^2) = p^j$ where $j \geq k - i$, so $b \in p^{k-i} q \mathbb{Z}_{p^k q^2}$ and $\left|p^{k-i} q \mathbb{Z}_{p^k q^2}\right| = \left|p^{k-i} q\{0,1,2,3, \ldots, p^k q^2 - 1\}\right| = \frac{p^k q^2}{p^{k-i} q} - 1$. Thus, since $\{p^{k-i} q \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$, $\deg(a) = p^i q - 1$.

Next, assume $i > \left\lfloor\frac{k-1}{2}\right\rfloor$. Let $a \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(a, p^k q^2) = p^i q^2$ and let $b \in Z(\mathbb{Z}_{p^k q^2})$ with gcd$(b, p^k q^2) = p^j q$, gcd$(b, p^k q^2) = p^j q^2$, gcd$(b, p^k q^2) = q$ and gcd$(b, p^k q^2) = q^2$ where a and b are adjacent if and only if $i + j \geq k$. Since gcd$(b, p^k q^2) = p^j q$, gcd$(b, p^k q^2) = p^j q^2$, gcd$(b, p^k q^2) = q$ and gcd$(b, p^k q^2) = q^2$ where $j \geq k - i$, so $b \in p^{k-i} q \mathbb{Z}_{p^k q^2}$ and $\left|p^{k-i} q \mathbb{Z}_{p^k q^2}\right| = \left|p^{k-i} q\{0,1,2,3, \ldots, p^k q^2 - 1\}\right| = \frac{p^k q^2}{p^{k-i} q} - 2$. Thus, since $\{p^{k-i} q \cdot 0\} \notin Z(\mathbb{Z}_{p^k q^2})$ and gcd$(p^i q, p^k q^2) = p^i q$ for $i > \left\lfloor\frac{k-1}{2}\right\rfloor$, so $\deg(a) = p^i q - 2$.

□

Since $\left|\mathbb{Z}_{p^k q^2}\right| = \left|\mathbb{Z}_{p^k}\right| \cdot \left|\mathbb{Z}_{q^2}\right|$, then the number of vertices in $\Gamma(\mathbb{Z}_{p^k q^2})$ for a given degree is divided into five cases, as given in Propositions 8, 9, 10, 11, and 12 where gcd$(a, p^k q^2) = p^i$, gcd$(a, p^k q^2) = q$, gcd$(a, p^k q^2) = q^2$, gcd$(a, p^k q^2) = p^i q$, and gcd$(a, p^k q^2) = p^i q^2$, respectively.

Proposition 8

Let $a \in Z(\mathbb{Z}_{p^k q^2})$ where gcd$(a, p^k q^2) = p^i$. Then,

$$\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = \begin{cases} (p^{k-i} - p^{k-(i+1)}) q(q-1), & \text{for } \; 1 \leq i \leq k-1, \\ p^{k-i} q(q-1), & \text{for } \; i = k. \end{cases}$$

Proof. Given $a \in Z(\mathbb{Z}_{p^k q^2})$ where gcd$(a, p^k q^2) = p^i$. Then,

a) $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = (p^{k-i} - p^{k-(i+1)})$ for $1 \leq i \leq k-1$ and $b \in Z(\mathbb{Z}_{p^k q^2})$ where gcd$(b, q^2) = q^2$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = q^2 - q = q(q-1)$. So $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = (p^{k-i} - p^{k-(i+1)}) q(q-1)$.

b) $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = p^{k-i}$ for $i = k$ and $b \in Z(\mathbb{Z}_{p^k q^2})$ where gcd$(b, q^2) = q^2$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = q^2 - q = q(q-1)$. So, $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = p^{k-i} q(q-1)$.

□

Proposition 9 Let $a \in Z(\mathbb{Z}_{p^k q^2})$ where gcd$(a, p^k q^2) = q$. Then, $\left|V\left(\Gamma(\mathbb{Z}_{p^k q^2})\right)\right| = (p^k - p^{k-1})(q-1)$.

Proof. Given $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = q$. Then, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (p^k - p^{k-1})$ and $b \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(b, p^kq^2) = q$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (q-1)$. So, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (p^k - p^{k-1})(q-1)$. □

Proposition 10 Let $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = q^2$. Then, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (p^k - p^{k-1})$.

Proof. Given $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = q^2$. Then, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (p^k - p^{k-1})$ and $b \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(b, q^2) = q$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = 1$. So, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (p^k - p^{k-1})(1) = (p^k - p^{k-1})$. □

Proposition 11 Let $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = p^iq$. Then,

$$\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \begin{cases} (p^{k-i} - p^{k-(i+1)})(q-1), & \text{for } 1 \le i \le k-1, \\ p^{k-i}(q-1), & \text{for } i = k. \end{cases}$$

Proof. Given $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = p^iq$. Then,

c) $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \left(p^{k-i} - p^{k-(i+1)}\right)$ for $1 \le i \le k-1$ and $b \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(b, q^2) = q^2$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (q-1)$. So, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \left(p^{k-i} - p^{k-(i+1)}\right)(q-1)$.

d) $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = p^{k-i}$ for $i = k$ and $b \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(b, q^2) = q^2$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = (q-1)$. So, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = p^{k-i}(q-1)$.

□

Proposition 12 Let $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = p^iq^2$. Then, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \left(p^{k-i} - p^{k-(i+1)}\right)$ for $1 \le i \le k-1$.

Proof. Given $a \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(a, p^kq^2) = p^iq^2$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \left(p^{k-i} - p^{k-(i+1)}\right)$ for $1 \le i \le k-1$ and $b \in Z(\mathbb{Z}_{p^kq^2})$ where $\gcd(b, q^2) = 1$, then $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = 1$. So, $\left|V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \left(p^{k-i} - p^{k-(i+1)}\right)(1) = \left(p^{k-i} - p^{k-(i+1)}\right)$. □

Theorem 1 The number of edges for $\Gamma(\mathbb{Z}_{p^kq^2})$,

$$\left|E\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \frac{1}{2}\left[(q-1)\left((p^k - p^{k-1})\left(q\left(2k - \frac{p^{k-1}-1}{p^k - p^{k-1}}\right) - \frac{p^{k-1}-1}{p^k - p^{k-1}} - \frac{p^{\left\lceil\frac{k-1}{2}\right\rceil}-1}{p^k - p^{k-1}}\right) + q(2p^k - 1) - 2\right) + (p^k - p^{k-1})\left(q^2(k-1) - \frac{p^{k-1}-1}{p^k - p^{k-1}} - \frac{p^{\left\lceil\frac{k-1}{2}\right\rceil}-1}{p^k - p^{k-1}}\right)\right].$$

Proof. According to the handshaking lemma, the sum of the degrees of all vertices in an undirected graph is twice the number of edges, as stated below:

$$\left|E\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \frac{1}{2}\sum_{a \in V\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)} deg(a).$$

Applying it from Proposition 3 to Proposition 12,

$$\left|E\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \frac{1}{2}\Bigg[q(q-1)\sum_{i=1}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)(p^i-1) + q(q-1)\sum_{i=1+k-1}^{k}p^{k-i}(p^i-1)$$
$$+(q-1)(p^k-p^{k-1})(q-1) + (p^k-p^{k-1})(q^2-1)$$
$$+(q-1)\sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(p^{k-i}-p^{k-(i+1)}\right)(p^iq-1) + (q-1)\sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(p^{k-i}-p^{k-(i+1)}\right)(p^iq^2-1)$$
$$+(q-1)\sum_{i=1+k-1}^{k}p^{k-i}(p^i-2) + \sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)(p^iq-2)$$
$$+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)(p^iq^2-2)\Bigg].$$

By using the summation rules and geometric sequences,

$$\left|E\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \frac{1}{2}\Bigg[(q-1)\left(q(p^k-p^{k-1})\left(\sum_{i=1}^{k-1}1-\sum_{i=1}^{k-1}p^{-i}\right)+q(p^k-1)+2(p^k-p^{k-1})q\right.$$
$$\left.+(p^k-p^{k-1})\left(q\sum_{i=1}^{k-1}1-\sum_{i=1}^{k-1}p^{-i}-\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}p^{-i}\right)+p^kq-2\right)$$
$$+(p^k-p^{k-1})\left(q^2\sum_{i=1}^{k-1}1-\sum_{i=1}^{k-1}p^{-i}-\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}p^{-i}\right)\Bigg].$$

Therefore, the number of edges for $\Gamma(\mathbb{Z}_{p^kq^2})$,

$$\left|E\left(\Gamma(\mathbb{Z}_{p^kq^2})\right)\right| = \frac{1}{2}\Bigg[(q-1)\left((p^k-p^{k-1})\left(q\left(2k-\frac{p^{k-1}-1}{p^k-p^{k-1}}\right)-\frac{p^{k-1}-1}{p^k-p^{k-1}}-\frac{p^{\left\lceil\frac{k-1}{2}\right\rceil}-1}{p^k-p^{k-1}}\right)+q(2p^k-1)\right.$$
$$\left.-2\right)+(p^k-p^{k-1})\left(q^2(k-1)-\frac{p^{k-1}-1}{p^k-p^{k-1}}-\frac{p^{\left\lceil\frac{k-1}{2}\right\rceil}-1}{p^k-p^{k-1}}\right)\Bigg].$$

□

The following example illustrates the zero divisor graph for the integers modulo ring $\mathbb{Z}_{36}$ based on the results.

Example 1 Suppose $p=2, k=2$ and $q=3$ for the zero divisor graph for the integers modulo ring $\mathbb{Z}_{36}$, $\Gamma(\mathbb{Z}_{36})$ as shown in Figure 1. The set of all zero divisors or the set of vertices in $\Gamma(\mathbb{Z}_{36})$ using Proposition 1 is $\{2,3,4,6,8,9,10,12,14,15,16,18,20,21,22,24,26,27,28,30,32,33,34\}$ consists of 23 elements. By Proposition 2, number of vertices is $3(2^{2-1}(3-1)+2^2)-1=23$. Next, the number of edges for $\Gamma(\mathbb{Z}_{36})$,

$$|E(\Gamma(\mathbb{Z}_{36}))| = \frac{1}{2}\left[(3-1)\left((2^2-2^{2-1})\left(3\left(2(2)-\frac{2^{2-1}-1}{2^2-2^{2-1}}\right)-\frac{2^{2-1}-1}{2^2-2^{2-1}}-\frac{2^{\left\lceil\frac{2-1}{2}\right\rceil}-1}{2^2-2^{2-1}}\right)+3(2(2^2)-1)-2\right)+(2^2-2^{2-1})\left(3^2(2-1)-\frac{2^{2-1}-1}{2^2-2^{2-1}}-\frac{2^{\left\lceil\frac{2-1}{2}\right\rceil}-1}{2^2-2^{2-1}}\right)\right]=46.$$

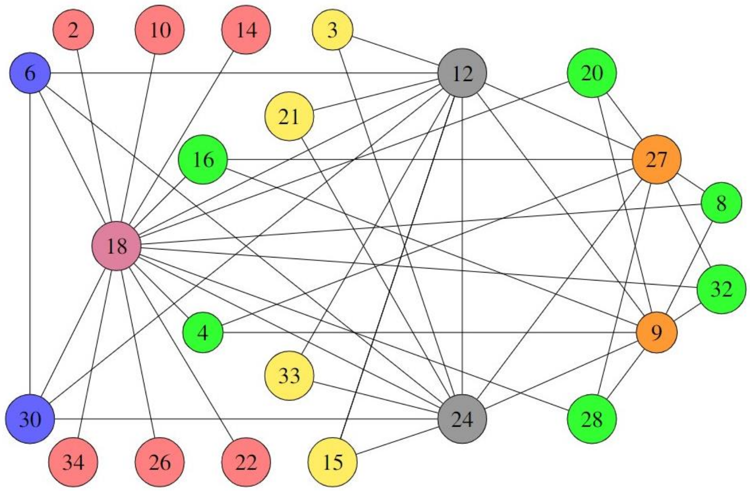

Figure 1 The illustration of $\Gamma(\mathbb{Z}_{36})$ with 23 vertices and 46 edges.

Example 2 Given that the zero divisor graph for the integers modulo ring $\mathbb{Z}_{75}$, $\Gamma(\mathbb{Z}_{75})$ when $p=3, k=1$ and $q=5$ as shown in Figure 2. The set of vertices in $\Gamma(\mathbb{Z}_{75})$ using Proposition 1 is $\{3,5,6,9,10,12,15,18,20,21,24,25,27,30,33,35,36,39,40,42,45,48,50,51,54,55,57,60,63,65,66,69,70,72\}$ consists of 34 elements. By Proposition 2, number of vertices is $5(3^{1-1}(5-1)+3^1)-1=34$. Next, the number of edges for $\Gamma(\mathbb{Z}_{75})$,

$$|E(\Gamma(\mathbb{Z}_{75}))| = \frac{1}{2}\left[(5-1)\left((3^1-3^{1-1})\left(5\left(2(1)-\frac{3^{1-1}-1}{3^1-3^{1-1}}\right)-\frac{3^{1-1}-1}{3^1-3^{1-1}}-\frac{3^{\left\lceil\frac{1-1}{2}\right\rceil}-1}{3^1-3^{1-1}}\right)+5(2(3^1)-1)-2\right)+(3^1-3^{1-1})\left(5^2(1-1)-\frac{3^{1-1}-1}{3^1-3^{1-1}}-\frac{3^{\left\lceil\frac{1-1}{2}\right\rceil}-1}{3^1-3^{1-1}}\right)\right]=86.$$

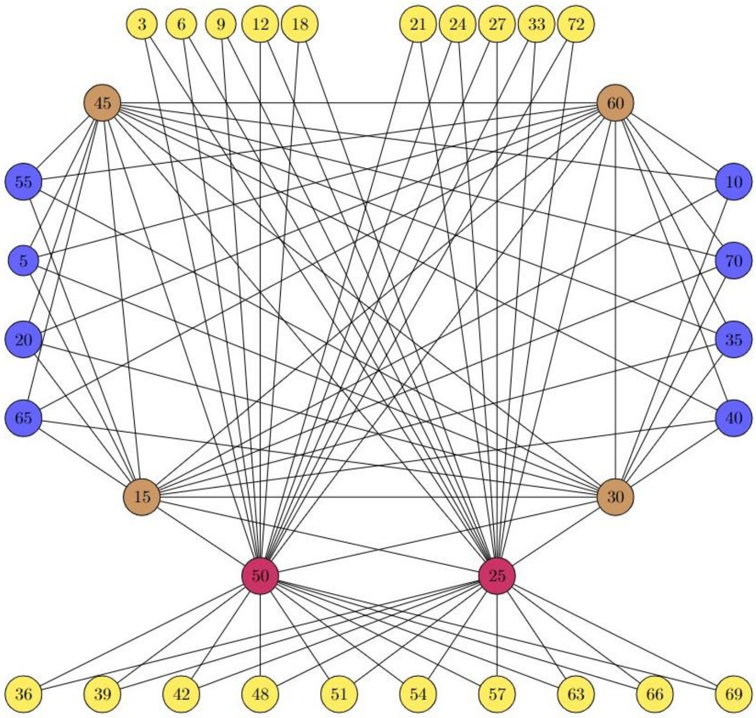

Figure 2 The illustration of $\Gamma(\mathbb{Z}_{75})$ with 34 vertices and 86 edges.

Conclusion

The set of all zero divisors and the number of zero divisors in the integers modulo ring $\mathbb{Z}_{p^k q^2}$ are determined as the new propositions. We also found the number of edges for the zero divisor graph of the integers modulo ring $\mathbb{Z}_{p^k q^2}$. Additionally, we provided some examples of the zero divisor graph for the ring to visualize the main findings.

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