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The Zagreb Indices of the Prime Power Cayley Graph of a Quaternion Group

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Abstract
Graph theory provides a powerful framework for understanding and analyzing relationships in various real-life scenarios, from social networks to transportation systems to biological interactions. A graph is a mathematical structure that consists of two sets which are a set of vertices (or nodes) and a set of edges connecting pairs of vertices. For example, in social networks like Facebook or Twitter, each person is a vertex in the graph, and connections between them (friendships, followers) are the edges. Graph theory helps these platforms suggest friends or recommend people to follow based on the connections in the graph. Meanwhile, a quaternion group is derived based on the quaternion number system, which extends the complex number. Quaternions are useful in calculations involving rotations in three dimensions, such as in three-dimensional computer graphics, computer vision, magnetic resonance imaging and crystallography. In this study, a variant of graph of groups namely the prime power Cayley graph of a quaternion group is constructed and its Zagreb index is determined. A Zagreb index is one of the topological indices which are used in chemical graph theory to characterize the structural features of molecular graphs. This index is calculated based on the degrees of vertices in a graph.

Keywords: Graph theory; Quaternion group; Topological indices; Zagreb index

1. Introduction

In graph theory, a graph is a mathematical structure that consists of two sets which are a set of vertices (or nodes) and a set of edges connecting pairs of vertices. Formally, a graph Γ is defined as an ordered pair $\Gamma = (V, E)$, where V is a non-empty set of vertices and E is a set of edges, where each edge is a 2-element subset of V [1]. In other words, an edge is a connection between two vertices.

Meanwhile, a group is defined as a set of elements and a binary operation applicable to any two elements within the set. This structure is defined by properties such as closure, associativity, the existence of an identity element, and the presence of inverses [2]. A quaternion group of order 2^n, Q_{2^n}, is represented by the following group presentation, with e denoting the identity element: $Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = e, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle$.

The concept of Cayley graphs, named after mathematician Arthur Cayley and introduced in 1878 [3], has been pivotal in the field of algebraic graph theory. A Cayley graph of a group G with respect to subset S of G is denoted as $Cay(G, S)$, provide a visual representation of group relationships through directed edges.

In 2022, the concept of Cayley graph was extended in [4] to prime power Cayley graph, associated with groups of order p^n, where p is a prime number and n is a positive integer.

The prime power Cayley graph which is denoted by $\widetilde{Cay}(G, S)$ is a graph where the set of vertices of the graph are elements of G, and two distinct vertices, g and h, are adjacent if $gh^{-1} \in S$ for the non-empty subset S of G containing element with prime power order.

While Cayley graphs offer a broad framework for visualizing group relationships, the focus shifts to prime power Cayley graphs when delving into groups of specific order, revealing intricate patterns and symmetries that uniquely arise from the prime power nature of the group. Cayley graphs provide a thorough framework for grasping group structures, but their use becomes more detailed when examining specific group orders.

The history of topological indices in graph theory unfolds as a response to the demand for quantitative descriptors of molecular structures and complex networks. The journey began in the mid-20th century when mathematicians sought mathematical tools to characterize chemical structures. Zagreb indices were introduced by Randić in the 1970s, offering insights into the complexity of molecular structures [5]. In subsequent decades, a surge of new indices emerged, addressing diverse aspects of molecular and network structures. In 2023, the Zagreb indices of power graphs of finite non-abelian groups were analyzed to understand their topological characteristics and structural invariants [6].

In this paper, the prime power Cayley graph of the quaternion group of order eight is constructed. Then, the Zagreb index of this graph is constructed.

2. Preliminaries

In this section, some basic concepts of group theory and graph theory which are used in this paper are included.

Definition 1 [7] (Generalized Quaternion Group)
For any $n \geq 3$, the generalized quaternion group Q_{2^n} of order 2^n is defined by

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = e, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle.$$

Definition 2 [4] (Graph of Group)
A graph of group G denoted as Γ_G consists of a finite nonempty set of objects called vertices and a set of unordered pairs of distinct vertices of Γ_G called edges. The vertex set of Γ_G denoted by $V(\Gamma_G)$ represents the elements of a group, while the edge set denoted by $E(\Gamma_G)$ represents the adjacency of the vertices.

Definition 3 [8] (Complete Graph)
A complete graph is a graph with order n, denoted as K_n is a graph where every distinct pair of vertices are adjacent.

Definition 4 [3] (Cayley Graph)
Let G be a group and S is a subset of $G \setminus \{e\}$. A Cayley graph of G relative to S is a graph such that the vertex of the graph is the elements of G and a vertex x is connected by a directed edge from x to y whenever $xy^{-1} = s$, for some $s \in S$. It is denoted by $Cay(G, S)$.

Definition 5 [4] (Prime Power Cayley Graph)
Let G be a group with $|G| = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ where p_i are primes for $a_i \in \mathbb{N}$. Let S be a non-empty subset of G in which $S = \{x \in G: |x| = p_i^{k_i}, 1 \leq k_i \leq a_i\}$ and $S = S^{-1} := \{s^{-1} | s \in S\}$. The prime power Cayley graph of G related to S, denoted by $\widetilde{Cay}(G, S)$ is a graph where the set of vertices of the graph are elements of G, and two distinct vertices, x and y, are adjacent if $xy^{-1} \in S$ for all $x, y \in G$.

Definition 6 [9] (Degree of a Vertex)
The degree of a vertex v which is denoted by d_v measures its connectivity by counting the number of edges incident to that vertex.

Definition 7 [10] (First Zagreb Index)
For a connected graph Γ,the first Zagreb index $M_1(\Gamma)$ is given by:

$$M_1(\Gamma) = \sum_{v \in V(\Gamma)} d_v^2,$$

where $V(\Gamma)$ is the set of vertices of Γ.

Definition 8 [10] (Second Zagreb Index)

For a connected graph Γ,the second Zagreb index $M_2(\Gamma)$ is given by:

$$M_2(\Gamma) = \sum_{u,v \in V(\Gamma)} d_u d_v ,$$

where u, v are adjacent vertices in Γ.

3. Research Methodology

Firstly, the group presentation of the quaternion group of order Q_8 is:

$$Q_8 = \langle a, b \mid a^4 = e, b^2 = a^2, b^{-1}ab = a^{-1} \rangle.$$

Based on the group presentation, the elements of Q_8 are listed as follows:

$$Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

Next, based on the group presentation and elements, the Cayley table of Q_8 is computed as in Table 3.1.

Table 3.1: Cayley Table of Q_8

*	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	a^3b	b	ab	a^2b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	ab	a^2b	a^3b	b
b	b	ab	a^2b	a^3b	a^2	a^3	e	a
ab	ab	a^2b	a^3b	b	a	a^2	a^3	e
a^2b	a^2b	a^3b	b	ab	e	a	a^2	a^3
a^3b	a^3b	b	ab	a^2b	a^3	e	a	a^2

Next, the prime power Cayley graph of Q_8 is constructed by referring to the Cayley table and definition of the graph. Finally, the first and second Zagreb index are computed by referring to the degrees of the vertices in the graph

4. Results and discussion

4.1. Construction of the Prime Power Cayley Graph of the Quaternion Group of Order Eight
By referring to the group presentation and Cayley table for Q_8 in Table 3.1, the order of each element in Q_8 is obtained as follows:

Table 4.1: Order of each element in Q_8

$g \in Q_8$	e	a	a^2	a^3	b	ab	a^2b	a^3b
$\|g\|$	1	4	2	4	4	2	2	2

Then, by using the definition of prime power Cayley graph given in Definition 5, the prime power Cayley graph of Q_8, $\widetilde{Cay}\,(Q_8, S)$ is associated to subset $S = \{x \mid |x| = 2, 2^2\} = \{a, a^2, a^3, b, ab, a^2b, a^3b\}$.

By Definition 5, $V\left(\widetilde{Cay}\,(Q_8, S)\right) = Q_8 = \{a, a^2, a^3, b, ab, a^2b, a^3b\}$. Let $g, h \in V\left(\widetilde{Cay}\,(Q_8, S)\right)$. Vertex g is adjacent to vertex h if $gh^{-1} \in S$ which implies that there exists $s \in S,$ such that $gh^{-1} = s$ or $g = sh$.

To find the set of edges of $\widetilde{Cay}\,(Q_8, S)$, firstly, consider $s \in S$.

By Definition 5, the vertex a is adjacent to a^2 since $a(a^2)^{-1} = a(a^2) = a^3$ in S. By using similar steps, the adjacencies between every pair of vertices are determined.

Therefore, the set of edges of $\widetilde{Cay}\,(Q_8, S)$ is

$$E\left(\widetilde{Cay}\,(Q_8,S)\right) = \{(a,e),(a,a^2),(a,a^3),(a,b),(a,ab),(a,a^2b),(a,a^3b),(a^2,e),(a^2,a^3),(a^2,b),(a^2,ab),(a^2,a^2b),(a^2,a^3b),(a^3,e),(a^3,b)(a^3,ab),(a^3,a^2b),(a^3,a^3b),(b,e),(b,ab),(b,a^2b),(b,a^3b),(ab,e),(ab,a^2b),(ab,a^3b),(a^2b,e),(a^2b,a^3b),(a^3b,e)\}.$$

Thus, the prime power Cayley graph, $\widetilde{Cay}\,(Q_8, S)$ where $S = \{a, a^2, a^3, b, ab, a^2b, a^3b\}$ is constructed as in Figure 4.1.

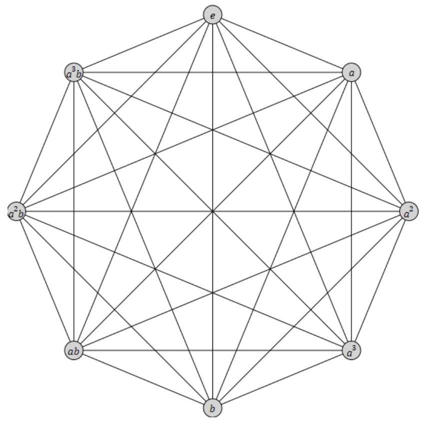

Figure 4.1 The Prime Power Cayley Graph $\widetilde{Cay}\,(Q_8, S)$

From Figure 4.1, it can be observed that the prime power Cayley graph of the quaternion group of order eight is a complete graph.

4.2 The Computation of First and Second Zagreb Indices

The prime power Cayley graph $\widetilde{Cay}\,(Q_8, S)$ is found to be a complete graph. Thus, $\widetilde{Cay}\,(Q_8, S) = K_8$. The degree of each vertex in $\widetilde{Cay}\,(Q_8, S)$ is equal to 7.

Therefore, by Definition 7, the first Zagreb index of $\widetilde{Cay}\,(Q_8, S)$ is:

$$\begin{aligned} M_1\left(\widetilde{Cay}\,(Q_8,S)\right) &= \sum_{v\in V\left(\widetilde{Cay}\,(Q_8,S)\right)} d_v^2 \\ &= (d_e)^2 + (d_a)^2 + (d_{a^2})^2 + (d_{a^3})^2 + (d_b)^2 + (d_{ab})^2 + (d_{a^2b})^2 + (d_{a^3b})^2 \\ &= 7^2 + 7^2 + 7^2 + 7^2 + 7^2 + 7^2 + 7^2 + 7^2 \\ &= 392\,. \end{aligned}$$

Then, by referring to Definition 8, the second Zagreb index of $\widetilde{Cay}\,(Q_8, S)$ is:

$$\begin{aligned} M_2\left(\widetilde{Cay}\,(Q_8,S)\right) &= \sum_{(u,v)\in E\left(\widetilde{Cay}\,(Q_8,S)\right)} d_u d_v \\ &= d_e d_a + d_e d_{a^2} + d_e d_{a^3} + d_e d_b + d_e d_{ab} + d_e d_{a^2b} + d_e d_{a^3b} + d_a d_{a^2} + d_a d_{a^3} + \\ &\quad d_a d_b + d_a d_{ab} + d_a d_{a^2b} + d_a d_{a^3b} + d_{a^2} d_{a^3} + d_{a^2} d_b + d_{a^2} d_{ab} + d_{a^2} d_{a^2b} + \\ &\quad d_{a^2} d_{a^3b} + d_{a^3} d_b + d_{a^3} d_{ab} + d_{a^3} d_{a^2b} + d_{a^3} d_{a^3b} + d_b d_{ab} + d_b d_{a^2b} + d_b d_{a^3b} \\ &= 28(7 \times 7) \end{aligned}$$

$$= 1372\,.$$

Conclusion

In this paper, the construction of the prime power Cayley graphs associated to the quaternion groups of order eight is presented. It is found that the graph is complete. Then, the first and second Zagreb index of this graph are computed based on the degrees of the vertices in the graph. Therefore, the first Zagreb index of $\widetilde{Cay}\,(Q_8, S)$ is 392 while the second Zagreb index of $\widetilde{Cay}\,(Q_8, S)$ is 1372.

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