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Laplacian Spectrum of the Deep Enhanced Power Graph of Generalized Quaternion Groups

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Abstract
This paper focuses on the Laplacian spectrum of the deep enhanced power graph of the generalized quaternion group, Q_{4n}. The Laplacian spectrum of a graph is the set of eigenvalues of its Laplacian matrix, which provides an important understanding of the graph's structural properties. For a finite group G, the deep enhanced power graph is defined as a simple undirected graph whose vertices represent all elements of G except for the nontrivial central element and two distinct vertices are adjacent if they belong to the proper cyclic subgroup. This research starts with the construction of the deep enhanced power graph for Q_{4n}, providing a general presentation of its structure, and continues with the derivation of its Laplacian matrix. The spectral properties of the Laplacian matrix are then analyzed, identifying the eigenvalues and their multiplicities. The result shows that there are five different multiplicities of eigenvalues in the Laplacian spectrum of the deep enhanced power graph of the generalized quaternion group.
Keywords: Laplacian spectrum, deep enhanced power graphs, graph of groups, and generalized quaternion groups

Introduction

Researchers can explore the complex relationships between group properties and graph invariants by defining graphs based on group elements and their connectivity. As a result, various types of graphs of groups have been introduced, including power graphs, commuting graphs, and enhanced power graphs. In the power graph, every vertex represents a group element, and an edge joins two vertices if one of them is a power of the other (Chattopadhyay & Panigrahi, 2015). Meanwhile, the commuting graph of a group is a simple graph with two vertices adjacent if they commute (Abdussakir et al., 2017). These two graphs inspired Aalipour et al. (2017) to combine those conditions by introducing the enhanced power graph. A cyclic graph is an alternative name for the enhanced power graph because it connects elements that generate the same cyclic subgroup (Ma et al., 2024). Another development of this graph is the deleted enhanced power graph, which excludes the identity element from the vertices (Costanzo et al., 2020; Costanzo & Lewis, 2021). The deep enhanced power graph is then introduced to further the exploration of the enhanced power graph by excluding nontrivial central elements from the vertex set (Mohamed et al., 2024).

Once the graphs of groups are introduced, it is natural to study their graph invariants. Laplacian spectrum of the graph is an example of the graph invariants that can be expressed as a set of sequences and polynomials. This spectrum refers to the multiplicities of the Laplacian matrix's eigenvalues derived from electrical circuit theory. For any graph Γ, the Laplacian matrix of Γ is formed by subtracting the adjacency matrix from the diagonal matrix of vertex degrees (Das, 2004). Their eigenvalues can reveal essential characteristics, notably graph connectivity and robustness. Numerous studies have discovered the Laplacian spectrum for different types of graphs of groups. For instance, Chattopadhyay and Panigrahi (2015) focused on the power graphs of cyclic and dihedral groups, followed by Panda (2019) on dicyclic groups and finite *p*-groups. For commuting graphs, Abdussakir et al. (2017), Dutta and Nath (2018), Torktaz and Ashrafi (2019), and Kumar et al. (2021) contributed to the development of this spectrum. Parveen et al. (2023) are the ones that focused on the Laplacian spectrum of the enhanced power graph of dihedral, semi-dihedral, and generalized quaternion groups. Despite the extensive studies conducted on the Laplacian spectrum, there are still opportunities to explore this scope.

This paper aims to find the Laplacian spectrum of the deep enhanced power graph of the generalized quaternion group. We construct the defined graph by considering all elements except the nontrivial central element, obtaining the neighbourhood of each vertex, and establishing its general presentation. This general presentation simplifies the method of obtaining the results of vertex degrees and the derivation of the Laplacian matrix. After analyzing the Laplacian matrix, the general formula for the Laplacian characteristic polynomial, which includes the eigenvalues and their multiplicities, is given, and it will contribute to the general Laplacian spectrum formula.

Preliminaries

This section presents some related information to establish the general presentation of the deep enhanced power graph of the generalized quaternion group and find its Laplacian spectrum. In this paper, a vertex set of Γ is denoted by $V(\Gamma)$ and an edge set is denoted by $E(\Gamma)$.

Definition 1 (Rotman, 2010) The generalized quaternion group of order $4n, Q_{4n}$, is a group generated by two elements a and b such that

$$Q_{4n} = \langle a, b : a^{2n} = b^4 = e, bab^{-1} = a^{-1}, a^n = b^2 \rangle,$$

for all $n \geq 2$.

Remark 1 The generalized quaternion group of order $4n$, can also be presented as $\langle a \rangle \cup \left(\sum_{j=0}^{2n-1} a^j b\right)$ where $\langle a \rangle = \{e, a, a^2, \cdots, a^{2n-1}\}$ and where $j \in \mathbb{N}$.

Proposition 1 (Conrad, 2014) The center of $Q_{4n}, Z(Q_{4n})$, is $\{e, a^n\}$ for all $n \geq 2$.

Remark 2 The identity element, e, refers to the trivial central element whereas a^n is considered as the nontrivial central element of Q_{4n}.

Definition 2 (West, 2001) A graph Γ is known as a complete graph of n vertices, K_n, provided that every vertex $v \in V(\Gamma)$ is connected to other vertices in $V(\Gamma)$.

Definition 3 (West, 2001) The neighbourhood of a vertex $v, N(v)$, is the set of all vertices adjacent to v in Γ.

Definition 4 (Bello, 2021) Let $\Gamma_1 = (V(\Gamma_1), E(\Gamma_1))$ and $\Gamma_2 = (V(\Gamma_2), E(\Gamma_2))$ be two subgraphs of Γ. The disjoint union of the graphs, $\Gamma_1 \cup \Gamma_2$ is a graph with the vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$ where $V(\Gamma_1)$ and $V(\Gamma_2)$ are disjoint sets of vertices. The join of the graphs, $\Gamma_1 + \Gamma_2$ is obtained from $\Gamma_1 \cup \Gamma_2$ by adding edges joining all vertices of Γ_1 to Γ_2.

Definition 5 (West, 2001) For any vertex $v \in V(\Gamma)$, the degree of a vertex v in Γ, $deg(v)$, is the number of edges that are incident on v.

Definition 6 (Mohamed et al., 2024) Let G be a finite group and V be all elements in G except the nontrivial central element of G. Deep enhanced power graph denoted as $\Gamma_{De}(G, V)$, is a simple undirected graph that has the elements of V as its vertices and two distinct vertices x and y are adjacent if and only if $\langle x, y\rangle$ is a proper cyclic subgroup of G.

Definition 7 (Jahanbani et al., 2021) The Laplacian matrix of a graph is $L(\Gamma) = D(\Gamma) - A(\Gamma)$ where $D(\Gamma)$ is a diagonal matrix containing vertex degrees and $A(\Gamma)$ is the adjacency matrix. The degree matrix is a diagonal matrix with the degree of each vertex on the diagonal. In $A(\Gamma) = (a_{ij})_{i,j=1}^{n}$, $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, otherwise $a_{ij} = 0$.

Definition 8 (Chattopadhyay & Panigrahi, 2015) The Laplacian characteristic polynomial of a graph Γ, $p_L(\lambda)$, is $p_L(\lambda) = \det(\lambda I - L(\Gamma))$ where λ is a real number, I is the identity matrix, and $L(\Gamma)$ is the Laplacian matrix.

Definition 9 (Yin, 2008) The Laplacian spectrum of a graph $\Gamma, L_s(\Gamma)$, refers to the Laplacian eigenvalues, $\lambda_1, \lambda_2, \cdots, \lambda_k$ with their multiplicities, $m_1, m_2, \cdots, m_k$, respectively and can be presented as $L_s(\Gamma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}$. These eigenvalues are derived from the roots of the characteristic polynomial, $p_L(\lambda) = \det(\lambda I - L(\Gamma)) = 0$.

Results and discussion

This section provides a general presentation of the deep enhanced power graph of Q_{4n}, $\Gamma_{De}(Q_{4n}, V)$ and its Laplacian spectrum. Firstly, information about the neighbourhood of each vertex is given to help establish the general presentation. Note that the neighbourhood of a vertex $v, N(v)$, is the set of all vertices adjacent to v in $\Gamma_{De}(Q_{4n}, V)$. Here, the deep enhanced power graph of Q_{12}, $\Gamma_{De}(Q_{12}, V)$, used as an example to help understand the overall concept.

Proposition 1 Let Q_{4n} be a generalized quaternion group of order $4n$ where $n \geq 2$. In $\Gamma_{De}(Q_{4n}, V)$,

i. $N(e) = V\backslash\{e\}$.
ii. $N(a^i) = \langle a\rangle\backslash\{a^i\}$ where $1 \leq i < 2n$ and $i \neq n$.
iii. $N(a^j b) = \langle a^j b\rangle\backslash\{a^j b\} = \{e, a^{j+n} b\}$ where $0 \leq j < 2n$.

Proof. Let Q_{4n} be a generalized quaternion group of order $4n$ where $n \geq 2$ and V be all elements in G except the nontrivial central element of G, a^n.

i. Since identity element, e is always adjacent to all other vertices in $\Gamma_{De}(Q_{4n}, V)$ because $\langle e, y\rangle = \langle y\rangle$ where $y \in V\backslash\{e\}$, therefore, $N(e) = Q_{4n}\backslash\{e, a^n\}$.
ii. Let $i \neq n$ and $1 \leq i < 2n$. Then, $N(a^i) = \langle a\rangle\backslash\{a^n\}$ since $\langle a\rangle \subseteq N(a^i)$. It is impossible for a^i to be adjacent to some $y \in Q_{4n}\backslash\langle a\rangle$ by the fact that a^i and y does not belong to the proper cyclic subgroup.
iii. Consider $a^j b$ where $0 \leq j < 2n$. By group relations, we have $a^{2n} = b^4 = e, bab^{-1} = a^{-1}, a^n = b^2$. Since the cyclic subgroup generated by $a^j b, \langle a^j b\rangle = \{e, a^n, a^j b, a^{j+n} b\}$, then, $N(a^j b) = \{e, a^{j+n} b\}$ for all $a^j b \in V$. □

Example 1 Let $Q_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2 b, a^3 b, a^4 b, a^5 b\}$ and the center of $Q_{12}, Z(Q_{12}) = \{e, a^3\}$. The vertices of $\Gamma_{De}(Q_{12}, V)$, $V = \{e, a, a^2, a^4, a^5, b, ab, a^2 b, a^3 b, a^4 b, a^5 b\}$, since a^3 is the nontrivial central element of Q_{12} and represent it as three partition sets that are $\{e\}$, $\{a, a^2, a^4, a^5\}$ and $\sum_{j=0}^{5} a^j b$. Using Proposition 1, the adjacency relations in $\Gamma_{De}(Q_{12}, V)$ can be determined, and they can be drawn as shown in Figure 1. The presentation of the graph is $\Gamma_{De}(Q_{12}, V) = K_1 + (K_4 \cup 3K_2)$.

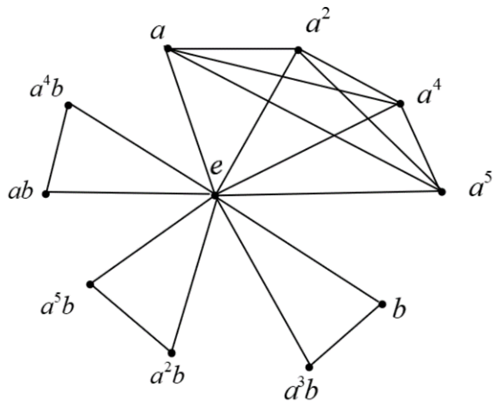

Figure 1 The deep enhanced power graph of Q_{12}

Theorem 1 Let Q_{4n} be a generalized quaternion group of order $4n$, where $n \geq 2$. Then,

$$\Gamma_{De}(Q_{4n}, V) = K_1 + \left(K_{2(n-1)} \cup nK_2\right).$$

Proof. Consider the generalized quaternion group of order $4n$, Q_{4n} where $n \geq 2$ and the vertex set of $\Gamma_{De}(Q_{4n}, V), V = Q_{4n} \backslash \{a^n\}$. There are three partitions set of V: $\{e\}$, $\langle a \rangle \backslash \{a^n\}$ and $\sum_{j=0}^{2n-1} a^j b$. From the neighbourhood information in Proposition 1, we can represent each of these partition sets. The set $\{e\}$ forms a subgraph K_1. Each $a^i \in \langle a \rangle \backslash \{a^n\}$ for $1 \leq i < 2n$ and $i \neq n$ is adjacent to all elements in the cyclic subgroup generated by a forming a complete subgraph $K_{2(n-1)}$. In $\sum_{j=0}^{2n-1} a^j b$, each $a^j b$ and $a^{j+n} b$ form a clique of size 2, resulting in n such cliques K_2. By Definition 4, we have $\Gamma_{De}(Q_{4n}, V) = K_1 + \left(K_{2(n-1)} \cup nK_2\right)$ as the general presentation since e is adjacent to all other elements in V, but the partition set $\langle a \rangle \backslash \{e\}$ is not connected to the partitions set $\sum_{j=0}^{2n-1} a^j b$. □

Next proposition describes the degrees of vertices corresponding to the elements of generalized quaternion group, which is also important in investigating the graph's spectral information.

Proposition 2 Let Q_{4n} be a generalized quaternion group of order $4n$, where $n \geq 2$. In $\Gamma_{De}(Q_{4n}, V)$,

i. $\deg(e) = 4n - 2$,
ii. $\deg(a^i) = 2n - 2$ where $1 \leq i < 2n$,
iii. $\deg(a^j b) = 2$ where $0 \leq j < 2n$.

Proof. Let Q_{4n} be a generalized quaternion group of order $4n$, where $n \geq 2$ and the vertex set of $\Gamma_{De}(Q_{4n}, V), V = Q_{4n} \backslash \{a^n\}$. The order of elements in $V = 4n - 1$. From Proposition 1 and Theorem 1, it is clear that

i. the identity element, e is adjacent to every vertex V, hence, $\deg(e) = 4n - 2$,
ii. for all $a^i \in V$ where $1 \leq i < 2n$, $\deg(a^i) = 2n - 2$ as reflect the interconnected with e and within the partition set $\langle a \rangle \backslash \{a^n\}$ of size $2n - 2$,
iii. for each $a^j b \in V$ where $0 \leq j < 2n$, $\deg(a^j b) = 2$ in which $a^j b$ are connected to e and $a^{j+n} b$.

□

Now, the Laplacian characteristic polynomial, $p_{L(\Gamma_{De})}$ and its spectrum of the deep enhanced power graph, $L_{spec}(\Gamma_{De})$ is provided. Note that Γ_{De} is an abbreviated form for $\Gamma_{De}(Q_{4n}, V)$.

Theorem 2 For all $n \geq 2$, the Laplacian characteristic polynomial of the deep enhanced power graph of Q_{4n} is given by

$$p_{L(\Gamma_{De})}(\lambda) = \lambda(\lambda - 1)^n(\lambda - 3)^n(\lambda - (2n - 1))^{2n-3}(\lambda - (4n - 1))$$

and its Laplacian spectrum is

$$L_{spec}(\Gamma_{De}) = \begin{pmatrix} 0 & 1 & 3 & 2n-1 & 4n-1 \\ 1 & n & n & 2n-3 & 1 \end{pmatrix}.$$

Proof. Using Theorem 1 and Proposition 2, the Laplacian matrix, $L(\Gamma_{De})$ of size $(4n-1) \times (4n-1)$ is constructed. The rows and columns of $L(\Gamma_{De})$ are indexed in order following the partition sets: $\{e\}$, $\langle a \rangle \backslash \{a^n\}$ and $\sum_{j=0}^{2n-1} a^j b$ as follows

$$L(\Gamma_{De}) = \begin{pmatrix} P_{(2n-1)\times(2n-1)} & Q_{(2n-1)\times 2n} \\ S_{2n\times(2n-1)} & T_{2n\times 2n} \end{pmatrix},$$

where,

$$P_{(2n-1)\times(2n-1)} = \begin{pmatrix} 4n-2 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 2n-2 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 2n-2 & \cdots & -1 & -1 \\ \vdots & \vdots & -1 & \ddots & \ddots & \vdots \\ -1 & -1 & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & \cdots & \cdots & -1 & 2n-2 \end{pmatrix}, Q_{(2n-1)\times 2n} = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$S_{2n\times(2n-1)} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 \end{pmatrix}, \quad T_{2n\times 2n} = \begin{pmatrix} A & 0 & \cdots & 0 \\ -0 & A & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}$$

with $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Using Definition 8, the Laplacian characteristic polynomial, $p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}(\lambda)$ can be written as $p_{L(\Gamma_{De})}(\lambda) = \det\left(\lambda I_{4n-1} - L(\Gamma_{De})\right)$. Thus,

$$p_{L(\Gamma_{De})}(\lambda) = \begin{vmatrix} \lambda I_{(2n-1)} - P_{(2n-1)\times(2n-1)} & Q_{(2n-1)\times 2n} \\ S_{2n\times(2n-1)} & \lambda I_{2n} - T_{2n\times 2n} \end{vmatrix}.$$

The desired Laplacian characteristic polynomial, $p_{L(\Gamma_{De})}(\lambda)$ is derived by applying elementary row operations to simplify the matrix towards its upper triangular form. These operations such as $R_k \leftarrow R_k - \frac{1}{\lambda-(4n-2)} R_1$ for $k = 2, 3, \cdots, 4n-1$ are systematically repeated for each subsequent row until the entire matrix is processed. Once the matrix is in upper triangular form, $p_{L(\Gamma_{De})}(\lambda)$ can be calculated as the product of the diagonal elements. Using Definition 9, their Laplacian spectrum, $L_{spec}(\Gamma_{De})$ holds, as stated. ◻

To have a clear view of Theorem 2, consider the Laplacian spectrum of the deep enhanced power graph of Q_{12} as an example.

Example 2 The deep enhanced power graph of Q_{12} is represented as $K_1 + (K_4 \cup 3K_2)$ and its Laplacian matrix, $L(\Gamma_{De})$ is of size 11×11 is indexed according to the partition sets as described in Example 1. Now, we have

$$L(\Gamma_{De}) = \begin{bmatrix} 10 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Then, we find the Laplacian characteristic polynomial,

$$p_{L(\Gamma_{De})}(\lambda) = \det\left(\lambda I_{4n-1} - L(\Gamma_{De})\right)$$
$$= \begin{vmatrix} \lambda-10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \lambda-4 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \lambda-4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \lambda-4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \lambda-4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \lambda-2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & \lambda-2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda-2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda-2 \end{vmatrix}.$$

First, eliminate the first column entries below the first row using $R_k \leftarrow R_k - \frac{1}{\lambda-10} R_1$ for $k = 2,3,\cdots,11$. Repeat similar operations for each subsequent column to eliminate entries below the diagonal until the matrix is completely transformed into upper triangular form. Therefore, we get

$$p_{L(\Gamma_{De})}(\lambda) = \lambda(\lambda-1)^3(\lambda-3)^3(\lambda-5)^3(\lambda-11).$$

Using Definition 9, the Laplacian spectrum, $L_{spec}(\Gamma_{De}) = \begin{pmatrix} 0 & 1 & 3 & 5 & 11 \\ 1 & 3 & 3 & 3 & 1 \end{pmatrix}$ that indicates the eigenvalues of the Laplacian matrix and their multiplicities.

Conclusion

This paper establishes the general presentation of $\Gamma_{De}(Q_{4n}, V)$ and finds its Laplacian spectrum. Through a detailed construction and analysis of the graph $\Gamma_{De}(Q_{4n}, V)$, we also provide the neighbourhood and the degree of each vertex. Our study found that the Laplacian spectrum, $L_{spec}(\Gamma_{De})$, indicates five distinct eigenvalues with their respective multiplicities. This spectrum provides significant insights into the structural properties and connectivity of the graph. Our findings contribute to a deeper understanding of the spectral characteristics of graphs of finite groups and provides opportunities for further investigations in both theoretical and applied graph theory.

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