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The Sombor Index of the Nilpotent Graph of Modulo Integer Numbers

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Abstract
In recent years, Gutman introduced the Sombor index, a graph invariant based on vertex degrees. This article defines the nilpotent graph Γ_R of the ring R, comprising vertices representing all elements in R. The authors aim to establish the general formulas for the Sombor index, reduced Sombor index, and average Sombor index on the nilpotent graph of the modulo integer ring. The method used in this research is the theoretical approach method. The results include the general formulas for these indices of graph on the modulo integer nilpotent graph.
Keywords: nilpotent graph; topological index; Sombor index

Introduction
The graph G is defined as $G = (V(G), E(G))$, consisting of a pair of non-empty set of vertices V and a set of edges E that connect pairs of vertices [1]. According to Nikmehr and Kojasteh, a graph is nilpotent over a ring R if all its vertices are elements of R and two distinct vertices are adjacent if and only if the product of the two distinct vertices is a nilpotent element [6]. The research conducted by Malik et al. [4] is based on the definition by Nikmehr and Kojasteh and they have successfully obtained the characteristics of nilpotent graphs over the ring of integers modulo any prime power.

In 2021, Ivan Gutman introduced the Sombor index, a degree-based measure in graph theory and a part of topological indices. It is defined as the sum of $\sqrt{\deg(\mathrm{u})^2 + \deg(\mathrm{v})^2}$ for all pairs of adjacent vertices (u, v), where the degree of a vertex u is $\deg(u)$ [2]. One of the recent research topics involves studying graph representations in algebraic structures. Another trending area in graph studies focuses on the indices or parameters used to measure various graph properties. Several researchers have conducted studies on topological indices in graphs, such as Husni et al.'s research discussing the Szeged index and Padmakar-Ivan index on nilpotent graphs over the ring of integers modulo a prime power [3] and Putra et al.'s research discussing the topological indices of power graphs on the ring of integers modulo n [7]. However, research on the representation of the Sombor index is still rare due to its novelty. Therefore, this research aims to provide a general formula for the Sombor index of nilpotent graphs over the ring of integers modulo n.

Given the research background, this study is intriguing as it can expand the knowledge of the Sombor index on nilpotent graphs over the ring of integers modulo n. Here are some terminologies that form the basis of this research.

Definition 1. [5] An element x in a ring R is said to be nilpotent if $x^r = 0$ for some $r \in \mathbb{N}$.

The set of all nilpotent elements of the ring R is denoted by $N(R)$. It can be observed that $N(R)$ forms an ideal of the ring R.

Definition 2. [5] The nilpotent graph of the ring R, denoted as Γ_R is a graph with a vertex set consisting of all the elements in the ring R. Two distinct vertices u and v are said to be adjacent if $u, v \in N(R)$.

Definition 3. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then, the Sombor index of G, denoted by $SO(G)$, is defined as the sum of all pairs of adjacent vertices of the square root of the sum of the squares of their degrees written as:

$$SO(G) = \sum_{u,v \in E(G)} \sqrt{\deg(u)^2 + \deg(v)^2},$$

where $\deg(u)$ is the degree of the vertex u and $\deg(v)$ is the degree of the vertex v [2].

A previous study stated that the nilpotent graph of the integer modulo group $\mathbb{Z}_n$ is a connected graph containing several star subgraphs and a complete graph [5]. This fact is stated in the following theorems.

Definition 4. [8] The reduced Sombor Index. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, then the reduced Sombor index of G, denoted by $SO_{red}(G)$ is as follows:

$$SO_{red}(G) = \sum_{u,v \in E(G)} \sqrt{(\deg(u) - 1)^2 + (\deg(v) - 1)^2}.$$

Definition 5. [8] The average Sombor Index. Let R be a graph with the vertex set $V(R)$ and edge set $E(R)$. Then, the average Sombor index of R, denoted by $SO_{avr}(R)$, is defined as follows:

$$SO_{avr}(R) = \sum_{u,v \in E(R)} \sqrt{\left(\deg(u) - \frac{2m}{n}\right)^2 + \left(\deg(v) - \frac{2m}{n}\right)^2}$$

with m being the number of edges and n being the number of vertices.

Methodology

The stages to complete this research using the theoretical approach method are:

1. Collecting relevant sources that can help solve the problems in the research, including literature analysis related to graphs, rings, and the sombor index.
2. Analyzing cases with specific group order petterns to identify the structure of the resulting nilpotent graphs. These petterns are obtained from solving various similar examples.
3. Formulating initial conjectures about the characteristics of the sombor index, reduced sombor index, and average sombor index in nilpotent graphs of integers modulo. Then, proving these conjectures mathematically to establish the characteristics of these indices. If the conjectures are proven, they will be made into theorems.
4. Finally, drawing conclusions from the made results obtained.

Results and Discussion

Based on the terminologies in the background above, the results of this research include the Sombor index, reduced Sombor index, and average Sombor index on nilpotent graphs over the ring of integers modulo.

Theorem 1. [5] If $\mathbb{Z}_n$ *is* the ring of integers modulo $n = 2^k$ where $k \in \mathbb{N}$, then $\Gamma_{\mathbb{Z}_n}$ contains 2^{k-1} star subgraphs $K_{1,2^{k-1}}$.

Theorem 2. [5] If $\mathbb{Z}_n$ is the ring of integers modulo $n = 2^k$ where $k \in \mathbb{N}$, then $\Gamma_{\mathbb{Z}_n}$ contains the complete subgraph $K_{2^{k-1}}$.

Moreover, the nilpotent elements are adjacent to all elements. Hence, all nilpotent elements form the complete subgraph as mentioned in Theorem 2. Additionally, the non-nilpotent elements are not adjacent to other non-nilpotent elements. Therefore, each non-nilpotent element, along with all nilpotent elements, forms a star subgraph, as mentioned in Theorem 1. To illustrate this, see the following figure.

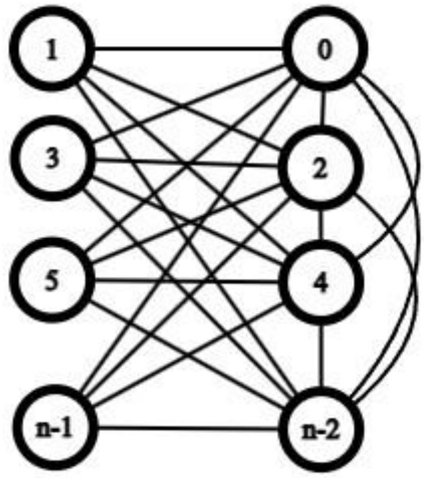

Figure 1 The nilpotent graphs of $\mathbb{Z}_n$ for $n = 2^k$

Example 1 If $n = 2^2$, then the graph diagram for the nilpotent graph of the ring $\mathbb{Z}_{2^2}$ shown in Figure 2.

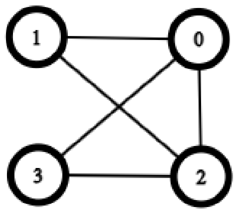

Figure 2 Nilpotent graph of $\mathbb{Z}_4$

Hence, based on Figure 2, we obtained $\deg(0) = \deg(2) = 3$, and $\deg(1) = \deg(3) = 2$. Then, the Sombor index of $SO(\Gamma_{\mathbb{Z}_4}) = 4\sqrt{3^2 + 2^2} + \sqrt{3^2 + 3^2} = 4\sqrt{13} + 3\sqrt{2}$.

Theorem 3. Let $\Gamma_{\mathbb{Z}_n}$ be a nilpotent graph of n, an integer modulo with $n = 2^k$, k is a natural number. Then, the Sombor index of the nilpotent graph is

$$SO(\Gamma_{\mathbb{Z}_n}) = (2^{2k-2})\sqrt{2^{2k-2} + (2^k - 1)^2} + \sqrt{2}(2^{k-1})(2^k - 2)(2^k - 1).$$

Proof. Given the graph $\Gamma_{\mathbb{Z}_n}$ with $n = 2^k, k \in \mathbb{N}$. We partitioned the graph $\Gamma_{\mathbb{Z}_n}$ into two partition: $V_1 = \{1, 3, 5, \dots, 2^k - 1\}$ and $V_2 = \{0, 2, 4, \dots, 2^k - 2\}$. Note that, V_1 contains all non-nilpotent elements, and V_2 contains all nilpotent elements and they form the complete subgraph mentioned in Theorem 2. Since every non-nilpotent element is adjacent to all nilpotent elements, and not adjacent to any non-nilpotent elements**,** we have that $\deg(u) = \frac{n}{2}, u \in V_1$. Since every nilpotent element is adjacent to all other elements, then $\deg(v) = n - 1, v \in V_2$. So, to prove this theorem, we divided it into two cases:

I. First case, we consider an edge uv where $u \in V_1$ and $v \in V_2$. Since $|V_1| = |V_2| = \frac{n}{2}$, then we have $\left(\frac{n}{2}\right)^2$ edges. Hence,

$$SO_1(\Gamma_{\mathbb{Z}_n}) = \left(\frac{n}{2}\right)^2 \sqrt{\left(\frac{n}{2}\right)^2 + (n-1)^2}$$
$$= (2^{2k-2})\sqrt{2^{2k-2} + (2^k - 1)^2}.$$

II. In the second case, we consider an edge uv where $u, v \in V_2$. Since $|V_2| = \frac{n}{2}$ and every element in V_2 is adjacent to all elements, then we have $\frac{n(n-1)}{2}$ edges. Hence,

$$SO_2(\Gamma_{\mathbb{Z}_n}) = \left(\frac{n(n-2)}{2}\right)\sqrt{(n-1)^2 + (n-1)^2}$$

$$= \left(\frac{n(n-2)}{2}\right)(n-1)\sqrt{2}$$
$$= \sqrt{2}(2^{k-1})(2^k-2)(2^k-1).$$

So, based on both cases, the Sombor index obtained from the graph $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$SO(\Gamma_{\mathbb{Z}_n}) = (2^{2k-2})\sqrt{2^{2k-2}+(2^k-1)^2} + \sqrt{2}(2^{k-1})(2^k-2)(2^k-1). \blacksquare$$

Theorem 4. Let $\Gamma_{\mathbb{Z}_n}$ be a nilpotent graph of n, an integer modulo with $n = 2^k$, k is a natural number. Then, the reduced Sombor index of the nilpotent graph is

$$SO_{red}(\Gamma_{\mathbb{Z}_n}) == \sqrt{5}(2^{2k-3})(2^k-2) + \sqrt{2}(2^{k-1})(2^k-2)^2.$$

Proof. Given the graph $\Gamma_{\mathbb{Z}_n}$ with $n = 2^k, k \in \mathbb{N}$. Similar to Theorem 3, we partitioned the graph $\Gamma_{\mathbb{Z}_n}$ into two partitions: $V_1 = \{1, 3, 5, \dots, 2^k - 1\}$ and $V_2 = \{0, 2, 4, \dots, 2^k - 2\}$.

I. First case, we consider an edge uv where $u \in V_1$ and $v \in V_2$. Since $|V_1| = |V_2| = \frac{n}{2}$, then we have $\left(\frac{n}{2}\right)^2$ edges. Hence, by definition 4

$$SO_{1_{red}}(\Gamma_{\mathbb{Z}_n}) = \left(\frac{n}{2}\right)^2 \sqrt{\left(\frac{n}{2}-1\right)^2 + \left((n-1)-1\right)^2}$$
$$= \frac{\sqrt{5}}{8} 2^{2k}(2^k-2)$$

II. In the second case, we consider an edge uv where $u, v \in V_2$. Since $|V_2| = \frac{n}{2}$ and every element in V_2 is adjacent to all elements, then we have $\frac{n(n-1)}{2}$ edges. Hence, by definition 4

$$SO_{2_{red}}(\Gamma_{\mathbb{Z}_n}) = \left(\frac{n(n-2)}{2}\right)\sqrt{\left((n-1)-1\right)^2 + \left((n-1)-1\right)^2}$$
$$= \frac{\sqrt{2}}{2} n(n-2)^2$$
$$= \frac{\sqrt{2}}{2} 2^k(2^k-2)^2.$$

Thus, based on both cases, the reduced Sombor index of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$SO_{red}(\Gamma_{\mathbb{Z}_n}) = \frac{\sqrt{5}}{8} 2^{2k}(2^k-2) + \frac{\sqrt{2}}{2} 2^k(2^k-2)^2$$
$$= \sqrt{5}(2^{2k-3})(2^k-2) + \sqrt{2}(2^{k-1})(2^k-2)^2. \blacksquare$$

Theorem 5. Let $\Gamma_{\mathbb{Z}_n}$ be a nilpotent graph of n, an integer modulo with $n = 2^k$, k is a natural number. Then the average Sombor index of the nilpotent graph is

$$SO_{avr}(\Gamma_{\mathbb{Z}_n}) = \frac{2^{2k}}{16}\sqrt{2^{2k+1} - 2^{k+4} + 2^3} + \frac{\sqrt{2}}{4}(2^{k-1})(2^k-2)^2.$$

Proof. Given the graph $\Gamma_{\mathbb{Z}_n}$ with $n = 2^k, k \in \mathbb{N}$. Similar to Theorem 3, we partitioned the graph $\Gamma_{\mathbb{Z}_n}$ into two partitions: $V_1 = \{1, 3, 5, \dots, 2^k - 1\}$ and $V_2 = \{0, 2, 4, \dots, 2^k - 2\}$. Noted that based on previous results, we have a total edge $\left(\frac{n}{2}\right)^2 + \left(\frac{n(n-1)}{2}\right) = \frac{3n^2-2n}{4}$, hence $\frac{m}{n} = \frac{3n-2}{4}$.

I. First case, we consider an edge uv where $u \in V_1$ and $v \in V_2$. Since $|V_1| = |V_2| = \frac{n}{2}$, then we have $\left(\frac{n}{2}\right)^2$ edges. Hence, by definition 5

$$SO_{avr}(\Gamma_{\mathbb{Z}_n}) = \left(\frac{n}{2}\right)^2 \sqrt{\left(\frac{n}{2} - \frac{3n-2}{4}\right)^2 + \left((n-1) - \frac{3n-2}{4}\right)^2}$$
$$= \frac{n^2}{4}\sqrt{\frac{2n^2 - 8n + 8}{16}}$$
$$= 2^{2k-4}\sqrt{2^{2k+1} - 2^{k+3} + 2^3}$$

II. Second case, we consider an edge uv where $u, v \in V_2$. Since $|V_2| = \frac{n}{2}$ and every element in V_2 is adjacent to all elements, then we have $\frac{n(n-1)}{2}$ edges. Hence, by definition 5

$$SO_{avr}(\Gamma_{\mathbb{Z}_n}) = \left(\frac{n(n-2)}{2}\right)\sqrt{\left((n-1) - \frac{3n-2}{4}\right)^2 + \left((n-1) - \frac{3n-2}{4}\right)^2}$$
$$= \left(\frac{n(n-2)}{2}\right)\left(\frac{n-2}{4}\right)\sqrt{2}$$
$$= \sqrt{2}(2^{k-3})(2^k - 2)^2.$$

Thus, based on both cases, the average Sombor index of $\Gamma_{\mathbb{Z}_n}$ is

$$SO_{avr}(\Gamma_{\mathbb{Z}_n}) = 2^{2k-4}\sqrt{2^{2k+1} - 2^{k+3} + 2^3} + \sqrt{2}(2^{k-3})(2^k - 2)^2. \blacksquare$$

Conclusion

This study results in a general formula for calculating the topological index of nilpotent graphs over the ring of integers modulo. Specifically, it finds the general formula for the Sombor index, the reduced Sombor index, and the average Sombor index. The results of this research have the potential to be used in various fields of mathematical and computer science studies involving graph analysis and algebraic structures, as well as providing a deeper understanding of the topological characteristics of nilpotent graphs over the ring of integers modulo.

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