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The p_i-Cayley Graph for Cyclic Group of Order p^2q^2

Athirah Zulkarnain[a]*, Nor Haniza Sarmin[a], Hazzirah Izzati Mat Hassim[a], Ahmad Erfanian[b]
[a]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[b]Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
*Corresponding author: athirah.zulkarnain91@gmail.com

Abstract
A Cayley graph is a graph constructed based on the subsets of a group. Different subsets of the group can produce various abstract structures for the Cayley graph. Some variations of Cayley graphs are defined by associating the graph with subsets of prime order elements or composite order elements of the group. Meanwhile, the p_i-Cayley graph has been introduced as a variation of the Cayley graph that is constructed using subsets with elements of prime power order for each prime dividing the order of the group. In this research, the p_i-Cayley graph is constructed for the cyclic group of order p^2q^2. Furthermore, some properties of the graph including the diameter and the chromatic number are determined.
Keywords: Cayley graph; cyclic group; diameter; chromatic number

Introduction
A Cayley graph was introduced by Arthur Cayley in 1878 [1]. The Cayley graph of a group is constructed based on the subsets of the group. The vertices of the Cayley graph are the elements of the group, and two distinct vertices, g and h are adjacent if gh^{-1} is in the subset.

In 2020, Konstantinova and Lytkina [3] explored the Cayley graph for the symmetric group and the alternating group. In the same year, Chen et al. [4] investigated the perfect codes of the Cayley graph for finite groups. In 2021, Zhang and Zhou [5] studied the subgroup perfect codes of the Cayley graph for finite groups. In 2022, Siemons and Zalesski [6] examined the Cayley graph of the symmetric group and the alternating group, focusing on the eigenvalues of the Cayley graph. In 2024, Movahedi [7] explored the energy of the Cayley graph for abelian groups.

Back in 2015 and 2019, Tolue introduced two variations of Cayley graphs, namely the prime order Cayley graph [8] and the composite order Cayley graph [9], respectively. The prime order Cayley graph is constructed by using subset with prime order elements only, while the composite order Cayley graph is constructed by using subset with composite order elements only.

Then, in 2020, Zulkarnain et al. [2] introduced another variation of Cayley graph, namely the p_i-Cayley graph. The p_i-Cayley graph is constructed by using subsets that contain prime power elements for each prime dividing the order of the group. In [2], the p_i-Cayley graph was constructed for alternating group and symmetric group on 4 letters. Hence, in this paper, the p_i-Cayley graph is constructed for a cyclic group of order p^2q^2, labeled as $C_{p^2q^2}$.

This paper is divided into four sections. The first section provides some research background on Cayley graphs. The second section contains the fundamental definitions in graph theory, including complete graphs and p_i-Cayley graphs, which are used throughout this research. The results are presented in the third section. Finally, the fourth section provides the conclusion.

Preliminaries

Some basic definitions in graph theory, including complete graphs and p_i-Cayley graphs, used in this study are stated in this section.

The definition of a complete graph is as follows:

Definition 1 [10] A complete graph of n-vertices, labeled as K_n, is a graph where each pair of distinct vertices is adjacent.

In this paper, the union of m-copies of complete graphs with n-vertices is labeled as mK_n.

Next, the p_i-Cayley graph associated with a group is defined.

Definition 2 [1] Let G be a group with $|G| = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_n^{k_n}$, where p_i are primes, and $k_i \in N$ for $i = 1,2,\ldots,n$. Define $S^{(p_i)} = \{g \in G \mid |g| = p_i^{r_i}, 1 \le r_i \le k_i\} \subseteq G$ with $S^{(p_i)} = {S^{(p_i)}}^{-1} \coloneqq \{s^{-1} | s \in S^{(p_i)}\}$. The p_i-Cayley graph associated to G with respect to $S^{(p_i)}$, denoted as $p_i - Cay\left(G, S^{(p_i)}\right)$, is a graph in which the vertices are the elements of G, and two distinct vertices, g and h, are adjacent if $gh^{-1} \in S^{(p_i)}$ for all $g, h \in G$.

Next, the results for the p_i-Cayley graph for a cyclic group of order p^2q^2 are presented.

Results and discussion

In this section, the p_i-Cayley graph for the cyclic group of order p^2q^2, labeled as $C_{p^2q^2}$, is explored. First, the elements of $C_{p^2q^2}$ generated by x are listed as follows:

$$C_{p^2q^2} = \{e, x, x^2, \ldots, x^{p^2q^2-1}\}.$$

Since there are two primes dividing the order of $C_{p^2q^2}$, namely p and q, then by Definition 2, two subsets are obtained, which are labeled as $S^{(p)}$ and $S^{(q)}$ as shown below:

1) The subset $S^{(p)}$ contains elements with orders p or p^2, which can be expressed as follows:
$$S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2 = \{x^{q^2}, x^{2q^2}, \ldots, x^{(p^2-1)q^2}\} = \{x^{mq^2} \mid 1 \le m \le p^2 - 1\}.$$

2) The subset $S^{(q)}$ contains elements with orders q or q^2, and it can be represented as:
$$S^{(q)} = \{g \in C_{p^2q^2} \mid |g| = q, q^2\} = \{x^{p^2}, x^{2p^2}, \ldots, x^{(q^2-1)p^2}\} = \{x^{kp^2} \mid 1 \le k \le q^2 - 1\}.$$

Hence, two p_i-Cayley graphs are formed with respect to each subset, labeled as $p - Cay\left(C_{p^2q^2}, S^{(p)}\right)$ and $q - Cay\left(C_{p^2q^2}, S^{(q)}\right)$. In order to construct the $p - Cay\left(C_{p^2q^2}, S^{(p)}\right)$ (given as Theorem 1), seven lemmas are first needed.

Lemma 1 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Define $A_i = \{x^{lq^2+i} \mid 1 \le l \le p^2 - 1\}$, $i = 1,2,\ldots,q^2$ and $B = \{x^j \mid 1 \le j \le q^2\} = \{x, x^2, \ldots, x^{q^2}\}$. Then, the set of elements in $C_{p^2q^2}$ is $B \cup \left(A_1 \cup A_2 \cup \ldots \cup A_{q^2}\right)$.

Proof Let $B \cup \left(A_1 \cup A_2 \cup \ldots \cup A_{q^2}\right) = C_{p^2q^2}$. If $g \in C_{p^2q^2}$, then $g = x^k$, for $1 \le k \le p^2q^2$. There are a few partitions that need to be checked. The partitions are divided into two. The first partition is

$1 \leq k \leq q^2$ and the second partition is $k = lq^2 + i, 1 \leq l \leq p^2 - 1$. If $1 \leq k \leq q^2$, then $g \in B$. If $k = lq^2 + i, 1 \leq l \leq p^2 - 1$, then $g \in A_i, 1 \leq i \leq q^2$. Therefore, $B \cup \left(A_1 \cup A_2 \cup \ldots \cup A_{q^2}\right)$ is the set of elements in $C_{p^2q^2}$ in $p - Cay\left(C_{p^2q^2}, S^{(p)}\right)$. ■

Lemma 2 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Let $A_i = \{x^{lq^2+i} \mid 1 \leq l \leq p^2 - 1\}$, $i = 1,2, \ldots, q^2$. Then, all vertices in each A_i of $p - Cay\left(C_{p^2q^2}, S^{(p)}\right)$ are adjacent to each other for $1 \leq i \leq q^2$.

Proof Let g and h be the vertices in A_i, where $g = x^{lq^2+i}$and $h = x^{l'q^2+i}$. Then, gh^{-1} is in $S^{(p)}$ since $|gh^{-1}| = \left|x^{(l-l')q^2}\right| = p^2$, which is shown in the following:

$$\left|x^{(l-l')q^n}\right| = \frac{p^2q^2}{\left((l-l')q^2, p^2q^2\right)} = \frac{p^2q^2}{q^2} = p^2.$$

Therefore, all vertices in each A_i for $1 \leq i \leq q^2$ are adjacent to each other. ■

Lemma 3 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Let $A_i = \{x^{lq^2+i}\}$ and $A_j = \{x^{l'q^2+j}\}$ for $i, j = 1,2, \ldots, q^2$ and $1 \leq l \leq p^2 - 1$. Then, the vertices in A_i and A_j of $p - Cay\left(C_{p^2q^2}, S^{(p)}\right)$ for $i \neq j$ are not adjacent to each other.

Proof Let $A_i = \{x^{lq^2+i} \mid 1 \leq l \leq p^2 - 1\}$ and $A_j = \{x^{l'q^2+j} \mid 1 \leq l' \leq p^2 - 1\}$ for $i \neq j$. Let $g = x^{lq^2+i}$ and $h = x^{l'q^2+j}$. The product of g and the inverse of h can be determined in two cases: the first case is when $l = l'$, and the second case is when $l \neq l'$.

Case 1: When $l = l'$ and $i \neq j, gh^{-1} = x^{i-j}$. Then x^{i-j} is not in $S^{(p)}$ since $\left|x^{i-j}\right|$ is not equal to p or p^2, as shown below:

$$\left|x^{i-j}\right| = \frac{p^2q^2}{(i-j, p^2q^2)} = \frac{p^2q^2}{1} = p^2q^2.$$

Therefore, if $l = l'$ and $i \neq j$, then $x^{i-j} \notin S^{(p)}$.

Case 2: When $l \neq l'$ and $i \neq j, gh^{-1} = x^{(l-l')q^2+(i-j)}$. Then $x^{(l-l')q^2+(i-j)}$ is not in $S^{(p)}$ since $\left|x^{(l-l')q^2(i-j)}\right|$ is not equal to p or p^2, as shown below:

$$\left|x^{(l-l')q^2+(i-j)}\right| = \frac{p^2q^2}{\left((l-l')q^2 + (i-j), p^2q^2\right)} = \frac{p^2q^2}{1} = p^2q^2.$$

Therefore, if $l \neq l'$ and $i \neq j$, then $x^{(l-l')q^2+(i-j)} \notin S^{(p)}$.

Based on these two cases, g and h are not adjacent, which implies that the vertices in different sets of $p - Cay(C_{p^2q^2})$, A_i and A_j, are not adjacent for $i \neq j$.

■

Lemma 4 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Let $B = \{x^j \mid 1 \leq j \leq q^2\} = \{x, x^2, \ldots, x^{q^2}\}$. Then, the vertices in B of $p - Cay(C_{p^2q^2})$ are not adjacent to each other.

Proof Let $g = x^i, h = x^j \in B$ for $1 \leq i \neq j \leq q^2$. Then, $(x^i)(x^j)^{-1} = x^{i-j} \notin S^{(p)}$ because $|x^{i-j}| = p^2q^2$ as shown below:

$$|x^{i-j}| = \frac{p^2q^2}{(i-j, p^2q^2)} = \frac{p^2q^2}{1} = p^2q^2.$$

Therefore, there is no adjacent vertices in set B. ■

Lemma 5 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Let $A_i = \{x^{lq^2+i} \mid 1 \leq l \leq p^2 - 1\}$, $i = 1,2, \dots, q^2$ and $B = \{x^j \mid 1 \leq j \leq q^2\} = \{x, x^2, \dots, x^{q^2}\}$. Then, in the $p - Cay(C_{p^2q^2}, S^{(p)})$, each vertex in B is adjacent to all vertices in the corresponding A_i where $i = j$.

Proof Let $A_i = \{x^{lq^2+i} \mid 1 \leq l \leq p^2 - 1\}$ for $i = 1,2, \dots, q^2$ and let $B = \{x^j \mid 1 \leq j \leq q^2\} = \{x, x^2, \dots, x^{q^2}\}$. Consider $g = x^{lq^2+i} \in A_i$ and $h = x^j \in B$. Then, for $i = j$, $gh^{-1} = (x^{lq^2+j})(x^j)^{-1} = x^{lq^2}$ belongs to $S^{(p)}$ since $|x^{lq^2}| = p^2$, as demonstrated below:

$$|x^{lq^2}| = \frac{p^2q^2}{(lq^2, p^2q^2)} = \frac{p^2q^2}{q^2} = p^2.$$

Therefore, $x^j \in B$ is adjacent to all vertices in respective A_i for $i = j$ in $p - Cay(C_{p^2q^2}, S^{(p)})$.

■

Lemma 6 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Let $A_i = \{x^{lq^2+i} \mid 1 \leq l \leq p^2 - 1\}$ for $i = 1,2, \dots, q^2$ and $B = \{x^j \mid 1 \leq j \leq q^2\} = \{x, x^2, \dots, x^{q^2}\}$. Then, in $p - Cay(C_{p^2q^2}, S^{(p)})$, $x^{lq^2+i} \in A_i$ and $x^j \in B$ for $i \neq j$ are not adjacent.

Proof Let $g = x^{lq^2+i} \in A_i$ and $h = x^j \in B$ for $1 \leq i \neq j \leq q^2$. Then, $gh^{-1} = x^{lq^2+i-j}$ which is not in $S^{(p)}$. This is because $|x^{lq^2+i-j}| = p^2q^2$ as shown below:

$$|x^{lq^2+i-j}| = \frac{p^2q^2}{(lq^2+i-j, p^2q^2)} = \frac{p^2q^2}{1} = p^2q^2.$$

Therefore, $x^{lq^2+i} \in A_i$ and $x^j \in B$ in $p - Cay(C_{p^2q^2}, S^{(p)})$ for $i \neq j$ is not adjacent to each other.

■

Lemma 7 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 generated by x. Let $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Let $A_i = \{x^{lq^2+i} \mid 1 \leq l \leq p^2 - 1\}$ for $i = 1,2, \dots, q^2$ and $B = \{x^j \mid 1 \leq j \leq q^2\} = \{x, x^2, \dots, x^{q^2}\}$. Then, a complete graph is formed in $p - Cay(C_{p^2q^2}, S^{(p)})$.

Proof Based on Lemma 2, there are $p^2 - 1$ vertices in each respective A_i that are adjacent to each other. Additionally, according to Lemma 5, one vertex in set B, denoted as x^j, is adjacent to these $p^2 - 1$ vertices in each respective A_i. Therefore, for every $i = j$, a component of a complete graph with p^2 vertices is formed between the $p^2 - 1$ vertices in each respective A_i and one vertex from set B.

■

Next, one of the main theorems in this paper is stated, followed by the proof.

Theorem 1 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 and $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\}$. Then,

$$p - Cay(C_{p^2q^2}, S^{(p)}) = q^2K_{p^2}.$$

Proof Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2. Thus, $C_{p^2q^2} = \{e, x, x^2, \dots, x^{p^2q^2-1}\}$. Let $S^{(p)}$ be a non-empty subset containing elements of order p, such that $S^{(p)} \coloneqq {S^{(p)}}^{-1}$. Thus, $S^{(p)} = \{g \in C_{p^2q^2} \mid |g| = p, p^2\} = \{x^{q^2}, x^{2q^2}, \dots, x^{(p^2-1)q^2}\}$.

The elements of $C_{p^2q^2}$ are the elements in the union of set B and set A_i for $1 \le i, j \le q^2$, as shown in Lemma 1. Then, based on Definition 2, the set of vertices for $p - Cay(C_{p^2q^2}, S^{(p)})$ is the set of elements in $C_{p^2q^2}$, that is, $V\left(p - Cay(C_{p^2q^2}, S^{(p)})\right) = p^2q^2$.

For the edges in the $p - Cay(C_{p^2q^2}, S^{(p)})$, two distinct vertices g and h in $V\left(p - Cay(C_{p^2q^2}, S^{(p)})\right)$ are adjacent if $gh^{-1} \in S^{(p)}$. In each respective A_i, the vertices are adjacent to each other based on Lemma 2. All these vertices in each respective set A_i are not adjacent to the vertices in set A_j for $i \neq j$, as shown in Lemma 3. This shows that each respective set of A_i forms a complete graph with $p^2 - 1$ vertices.

Then, based on Lemma 4, none of the vertices in set B are adjacent to each other. However, each vertex in set B, labeled as x^j, is adjacent to all vertices in their respective sets A_i when $i = j$, as proven in Lemma 5. The vertex $x^j \in B$ is not adjacent to the vertices in any other respective set A_i when $i \neq j$, as demonstrated in Lemma 6. Hence, a component of a complete graph with p^2 vertices is formed between each vertex in set B, denoted as x^j, and all vertices in their respective sets A_i when $i = j$, as proven in Lemma 7. The $p^2 - 1$ vertices come from each respective set A_i, and one vertex comes from set B. Therefore, it is proven that the total number of vertices in each component of the complete graph is p^2. Since there are q^2 vertices in set B, there are q^2 copies of complete graphs, where each complete graph contains p^2 vertices.

Therefore, the p-Cayley graph of $C_{p^2q^2}$ with respect to $S^{(p)}$ is the union of q^n copies of a complete graph, where each complete graph contains p^n vertices, namely $p - Cay(C_{p^2q^2}, S^{(p)}) = q^2K_{p^2}$.

■

Lastly, the p_i-Cayley graph with respect to $S^{(q)}$ is stated in the following theorem. The proof is similar to that of Theorem 1.

Theorem 2 Let $C_{p^2q^2}$ be a cyclic group of order p^2q^2 and $S^{(q)} = \{g \in C_{p^2q^2} \mid |g| = q, q^2\}$. Then,

$$q - Cay(C_{p^2q^2}, S^{(q)}) = p^2K_{q^2}.$$

Conclusion

In this paper, the p_i-Cayley graph of cyclic group of order p^2q^2 is found for each p_i dividing the order of the group. There are two subsets obtained, hence there are two p_i-Cayley graphs formed which are $p - Cay(C_{p^2q^2}, S^{(p)}) = q^2K_{p^2}$ and $q - Cay(C_{p^2q^2}, S^{(q)}) = p^2K_{q^2}$.

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