Proceedings of Science and Mathematics
Faculty of science, universiti teknologi malaysia
https://science.utm.my/procscimath/
Volume 26 (2024) 6-12

Sombor Energy of the Zero Divisor Graph for Some Ring of $\mathbb{Z}_{3^k}$

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Abstract
Sombor energy is a graph invariant computed from the Sombor matrix associated with a graph, is defined as the sum of the absolute values of its eigenvalues where the eigenvalues are computed based on Sombor matrix. A matrix representation whose elements are defined by the degrees of the vertices and the existence of edges in the graph is called the Sombor matrix. Meanwhile, the zero divisor graph is defined as a graph whose vertex set contains all of the non-zero zero divisors. If the product of two unique elements in the zero divisors equals zero, then they are adjacent. In this paper, the Sombor energy of the zero divisor graph for the commutative ring of integers modulo $3,$ where $k = 2$ and 3 are computed. In addition, the zero divisor graphs and their Sombor matrix are also presented.
Keywords: Sombor energy; zero divisor graph; graph theory; ring theory

Introduction
The energy of a graph serves as a fundamental measure in both graph theory and chemistry, encapsulating structural insights and chemical properties. Initially defined by Gutman [1], this concept emerged from the spectral analysis of graphs, defining the energy of the graph as the sum of all positive eigenvalues derived from its adjacency matrix, constituting the graph's spectrum [1]. This notion found inspiration from the Hückel Molecular Orbital (HMO) Theory, formulated in the 1930s to approximate energies related to π-electron orbitals in conjugated hydrocarbon molecules. Recently, in chemical graph theory, Gutman [2] introduced a new topological index based on vertex degrees, known as the Sombor index. Then, Gowtham and Narasimha [3] expanded the study by defining a new type of graph energy called Sombor energy and presenting the Sombor matrix for a graph.

Interestingly, the Sombor energy of some compounds has been found to correlate well with their total π-electron energy which it demonstrates the usefulness and applicability of this concept in chemical graph theory [3]. Since then, many researchers have expressed interest in this topic. For instance, Ghanbari [4] computed the Sombor energy for some graph classes, Rather and Imran [5] established the bounds on the Sombor energy of some graphs, Tabassum et al. [6] analysed the relationship between Sombor energy with another energy and Romdhini and Nawawi [7] presented the Sombor energy of non-commuting graph for dihedral group.

In addition, graph theory, in particular zero divisor graphs, has attracted a lot of attention because of its application to algebraic structures. Beck [8] initially introduced the idea of the zero divisor graph, primarily concentrating on colorings of commutative rings. Then, Anderson and Livingston [9] expanded on this work and offered a slightly altered definition for the zero divisor graph of commutative rings. In this paper, the definition proposed by Anderson and Livingston is adopted as the reference. There are plenty of research employing rings and graphs that have been conducted by earlier researchers like [10-12].

This paper focuses on Sombor energy of zero divisor graph for some ring of $\mathbb{Z}_{3^k}$ The first section of this paper is introduction, followed by preliminaries where some basic concepts and definition on ring theory, graph theory and energy are stated. In the last section, the main results are presented.

Preliminaries

In this section, some basic concepts and definitions in ring theory, graph theory and energy which are used to prove the main results are presented. The zero divisor graph are constructed by using Python software.

Definition 1 [13] Ring
A nonempty set with the two operations $+,\ \cdot:\ R\times R\rightarrow R$ is called a ring $R.$ It has the following properties. For any $a,\ b,\ c\in R,$

i. $\langle R,+\rangle$ forms an Abelian group;
ii. $\langle R,\cdot\ \rangle$ is associative, meaning $a\cdot\left(b\cdot c\right)=\left(a\cdot b\right)\cdot c;$
iii. The distributive laws hold: $\left(a+b\right)\cdot c=\left(a\cdot c\right)+\left(b\cdot c\right)$ and $a\cdot\left(b+c\right)=\left(a\cdot b\right)+\left(a\cdot c\right)$.

Definition 2 [13] Commutative Ring
A ring R is commutative if $ab=ba$ for all $a,b\in R.$

Definition 3 [13] Zero Divisors of a Ring
In a ring $R,$ when two nonzero elements a and b multiply together to give zero, that is $ab=0,$ these elements are called zero divisors. $Z(R)$ represents the set of all zero divisors in $R.$

Definition 4 [9] Zero Divisor Graph of Commutative Rings
Let $\Gamma(Z(R))$ be the zero divisor graph of commutative rings $R.$ The graph consists of non-zero zero divisors of R as the set of vertices and vertices a and b are adjacent if and only if $ab=ba=0.$

Definition 5 [14] Degree of a Vertex
A vertex's degree, represented as $\deg(v),$ is the total number of edges that are incident to $v.$

In order to compute the Sombor energy of a graph, the Sombor matrix of the zero divisor graph need to be first determined. Then, its eigenvalues from the Sombor polynomial are found. The following definition states the definition of Sombor matrix.

Definition 6 [3] Sombor Matrix
The Sombor matrix of a graph $\Gamma,\ SO(\Gamma),$ with vertex set, $V(\Gamma),$ and edge set, $E(\Gamma),$ is defined such that $SO_{ii}=0,\ \ SO_{ij}=\sqrt{\deg(i)^2+\deg(j)^2}$ if vertices v_i and v_j are adjacent, and $SO_{ij}=0$ otherwise, where $\deg(i)$ and $\deg(j)$ are the degrees of vertices v_i and $v_j,$ respectively.

Definition 7 [3] Sombor Polynomial
For a graph Γ, the Sombor polynomial, $P_{SO(\Gamma)}(\lambda)$ is defined as:

$$P_{SO(\Gamma)}(\lambda)=\left|\lambda I-SO\left(\Gamma\right)\right|,$$

where $SO(\Gamma)$ is the Sombor matrix of the graph $\Gamma,\ I$ is the identity matrix, and λ is a variable.

Definition 8 [3] Sombor Energy of a Graph
For a given graph Γ, the Sombor matrix $SO(\Gamma)$ is a real symmetric matrix, which means all the eigenvalues of this matrix are real numbers. These eigenvalues can be ordered from largest to smallest as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. The Sombor energy, $SE(\Gamma)$, of the graph is then defined as the total sum of the absolute values of these eigenvalues:

$$SE(\Gamma) = \sum_{i=1}^{n} |\lambda_i|.$$

The computation of the Sombor energy of a graph is illustrated in the following example.

Example 1 A graph such as $\Gamma = (V(\Gamma), E(\Gamma))$ where $V(\Gamma) = \{a,b,c,d,e\}$ is shown in Figure 1.

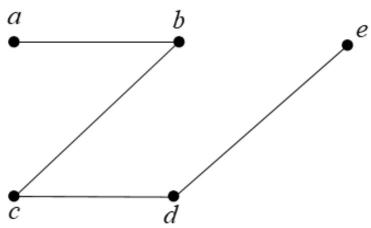

Figure 1 A graph consist of five vertices and four edges.

From Figure 1 and based on the Definition 6, the degrees of the vertices are $\deg(a)=1$, $\deg(b)=2$, $\deg(c)=2$, $\deg(d)=2$ and $\deg(e)=1$. Since the graph Γ has edges (a,b), (b,c), (c,d) and (d,e), the corresponding adjacent vertices will have the entry $\sqrt{\deg(i)^2+\deg(j)^2}$. Non-adjacent vertices will have the entry 0. For example, the entries for the elements a with b and also b with a are $\sqrt{1^2+2^2}=\sqrt{5}$. This gives the following symmetric Sombor matrix:

$$SO(\Gamma) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & \sqrt{5} & 0 & 0 & 0 \\ \sqrt{5} & 0 & 2\sqrt{2} & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} & 0 & \sqrt{5} \\ 0 & 0 & 0 & \sqrt{5} & 0 \end{bmatrix} \end{array}$$

Then, by Definition 7, the Sombor polynomial and the eigenvalues is calculated as follows.

$$\begin{aligned} P_{SO(\Gamma)} &= \det(\lambda I - SO(\Gamma)) \\ &= \begin{vmatrix} \lambda & -\sqrt{5} & 0 & 0 & 0 \\ -\sqrt{5} & \lambda & -2\sqrt{2} & 0 & 0 \\ 0 & -2\sqrt{2} & \lambda & -2\sqrt{2} & 0 \\ 0 & 0 & -2\sqrt{2} & \lambda & -\sqrt{5} \\ 0 & 0 & 0 & -\sqrt{5} & \lambda \end{vmatrix} \\ &= \lambda^5 - 26\lambda^3 + 105\lambda \end{aligned}$$

Thus, the Sombor polynomial $P_{S(\Gamma)}(\lambda)=0$ will give the eigenvalues $\lambda_1=0$, $\lambda_2=\sqrt{5}$, $\lambda_3=-\sqrt{5}$, $\lambda_4=\sqrt{21}$, $\lambda_5=-\sqrt{21}$. Hence, the Sombor energy of the graph,

$$\begin{aligned} SE(\Gamma) &= |0|+\left|\sqrt{5}\right|+\left|-\sqrt{5}\right|+\left|\sqrt{21}\right|+\left|-\sqrt{21}\right| \\ &= 2\sqrt{5}+2\sqrt{21} \\ &= 13.368. \end{aligned}$$

Results and Discussion

In this section, the Sombor energy of zero divisor graph for $\mathbb{Z}_{3^k}$ $k=2$ and 3 are shown in the main theorems.

Theorem 1 Let Γ be the zero divisor graph for $\mathbb{Z}_9$ which is $\mathbb{Z}_{3^2}$ Then, the Sombor energy of zero divisor graph for $\mathbb{Z}_9$, $SE(\Gamma(\mathbb{Z}_9)) = 2\sqrt{2}$.

Proof Let Γ be the zero divisor graph of $Z_9=\{0,1,2,3,4,5,6,7,8\}$. The non-zero zero divisors of Z_9 are 3 and 6 since $3\times 3=9\equiv 0$ (mod 9), $3\times 6=18\equiv 0$ (mod 9) and $6\times 6=36\equiv 0$ (mod 9). Hence, the vertex set of Γ is $\{3,6\}$. The zero divisor graph of $\mathbb{Z}_9$ is shown in Figure 2 below.

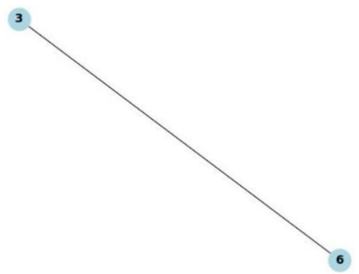

Figure 2 Zero divisor graph of $\mathbb{Z}_9$

Then, by the definition of Sombor matrix, the degrees of the vertices are $\deg(3)=\deg(6)=1$. Since the graph Γ has only one edge $(3,6)$, the Sombor matrix is:

$$SO(\Gamma)=\begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$$

To find the eigenvalues of the Sombor matrix $SO(\Gamma)$, the Sombor polynomial is solved by $\det(\lambda I - SO(\Gamma))=0$:

$$\det\begin{bmatrix} \lambda & -\sqrt{2} \\ -\sqrt{2} & \lambda \end{bmatrix} = (\lambda)(\lambda)-\left(-\sqrt{2}\right)\left(-\sqrt{2}\right)=\lambda^2-2=0$$

Then, solving $\lambda^2-2=0$, the eigenvalues of the Sombor matrix $SO(\Gamma)$ are $\lambda_1=\sqrt{2}$ and $\lambda_2=-\sqrt{2}$. As defined in Definiton 1, the Sombor energy is:

$$\begin{aligned} SE(\Gamma) &= \left|\sqrt{2}\right|+\left|-\sqrt{2}\right| \\ &= \sqrt{2}+\sqrt{2} \\ &= 2\sqrt{2}. \end{aligned}$$

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Theorem 2 Let Γ be the zero divisor graph for $\mathbb{Z}_{27}$ that is $\mathbb{Z}_{3^3}$ Then, the Sombor energy of zero divisor graph for commutative ring of order 27, $SE(\Gamma) = 61.2998$.

Proof Let Γ be the zero divisor graph of $\mathbb{Z}_{27} = \{0,1,2,3,\ldots,26\}$. The non-zero zero divisors of $\mathbb{Z}_{27}$ are $3,6,9,12,15,18,21$ and 24 since $3\times 9 \equiv 0$ (mod 27), $3\times 18 \equiv 0$(mod 27), $6\times 9 \equiv 0$(mod 27), $6\times 18 \equiv 0$(mod 27), $9\times 9 \equiv 0$(mod 27), $9\times 12 \equiv 0$(mod 27), $9\times 15 \equiv 0$(mod 27), $9\times 18 \equiv 0$(mod 27), $9\times 21 \equiv 0$(mod 27), $9\times 24 \equiv 0$(mod 27), $12\times 18 \equiv 0$(mod 27), $18\times 18 \equiv 0$(mod 27), $18\times 21 \equiv 0$(mod 27), $18\times 24 \equiv 0$(mod 27) and $15\times 18 \equiv 0$(mod 27). Hence, the vertex set of Γ is $\{3,6,9,12,15,18,21,24\}$. The zero divisor graph of $\mathbb{Z}_{27}$ is shown in Figure 2 below.

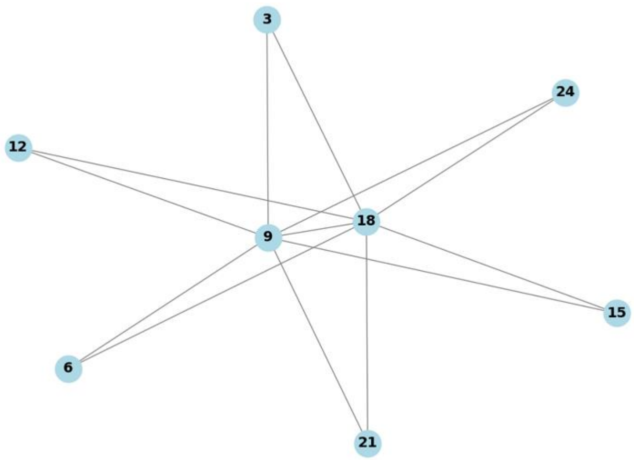

Figure 2 Zero divisor graph of $\mathbb{Z}_{27}$

Then, by the definition of Sombor matrix, the degrees of the vertices are $\deg(3)=2, \deg(6)=2$, $\deg(9)=7, \deg(12)=2, \deg(15)=2, \deg(18)=7, \deg(21)=2$ and $\deg(24)=2$. Since the graph Γ has edges of $(3,9),(3,18),(6,9),(6,18),(9,12),(9,15),(9,18),(9,21),(9,24),(12,18),(18,21),(18,24)$ and $(15,18)$ the Sombor matrix is:

$$SO(\Gamma)=\begin{bmatrix} 0 & 0 & \sqrt{53} & 0 & 0 & \sqrt{53} & 0 & 0 \\ 0 & 0 & \sqrt{53} & 0 & 0 & \sqrt{53} & 0 & 0 \\ \sqrt{53} & \sqrt{53} & 0 & \sqrt{53} & \sqrt{53} & \sqrt{98} & \sqrt{53} & \sqrt{53} \\ 0 & 0 & \sqrt{53} & 0 & 0 & \sqrt{53} & 0 & 0 \\ 0 & 0 & \sqrt{53} & 0 & 0 & \sqrt{53} & 0 & 0 \\ \sqrt{53} & \sqrt{53} & \sqrt{98} & \sqrt{53} & \sqrt{53} & 0 & \sqrt{53} & \sqrt{53} \\ 0 & 0 & \sqrt{53} & 0 & 0 & \sqrt{53} & 0 & 0 \\ 0 & 0 & \sqrt{53} & 0 & 0 & \sqrt{53} & 0 & 0 \end{bmatrix}$$

To find the eigenvalues of the Sombor matrix $SO(\Gamma)$, the Sombor polynomial is solved by $\det(\lambda I - SO(\Gamma)) = 0$:

$$\det\begin{bmatrix} \lambda & 0 & -\sqrt{53} & 0 & 0 & -\sqrt{53} & 0 & 0 \\ 0 & \lambda & -\sqrt{53} & 0 & 0 & -\sqrt{53} & 0 & 0 \\ -\sqrt{53} & -\sqrt{53} & \lambda & -\sqrt{53} & -\sqrt{53} & -\sqrt{98} & -\sqrt{53} & -\sqrt{53} \\ 0 & 0 & -\sqrt{53} & \lambda & 0 & -\sqrt{53} & 0 & 0 \\ 0 & 0 & -\sqrt{53} & 0 & \lambda & -\sqrt{53} & 0 & 0 \\ -\sqrt{53} & -\sqrt{53} & -\sqrt{98} & -\sqrt{53} & -\sqrt{53} & \lambda & -\sqrt{53} & -\sqrt{53} \\ 0 & 0 & -\sqrt{53} & 0 & 0 & -\sqrt{53} & \lambda & 0 \\ 0 & 0 & -\sqrt{53} & 0 & 0 & -\sqrt{53} & 0 & \lambda \end{bmatrix} = 0$$

Then, the eigenvalues of the Sombor matrix $SO(\Gamma)$ are calculated and found to be $\lambda_1 = 30.6499$, $\lambda_2 = -9.8995$, $\lambda_3 = -20.7504$, $\lambda_4 = 0$, $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_7 = 0$ and $\lambda_8 = 0$. As defined in Definiton 1, the Sombor energy is:

$$\begin{aligned} SE(\Gamma) &= |30.6449| + |-9.8995| + |-20.7504| \\ &= 30.6449 + 9.8995 + 20.7504 \\ &= 61.2998. \end{aligned}$$

■

Conclusion

In conclusion, the Sombor energy of the zero divisor graph for ring $\mathbb{Z}_{3^k}$ which $k = 2$ and 3 are computed in this paper. The zero divisor graph of $\mathbb{Z}_9$ and $\mathbb{Z}_{27}$ are also constructed by using Python software. Then the Sombor matrix and its eigenvalues are determined in order to compute its Sombor energy. As a result, the Sombor energy of the zero divisor graph of $\mathbb{Z}_{3^k}$ shows an increase with the order of the graph.

References

[1] Gutman, I. (1978). The energy of a graph. *Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz, 103*, 1-22.

[2] Gutman, I. (2021). Geometric approach to degree–based topological indices: Sombor indices. *Faculty of Science, University of Kragujevac*.

[3] Gowtham, K. J., & Narasimha, S. N. (2021). On Sombor energy of graphs. *Nanosystems: Physics, Chemistry, Mathematics, 12*(4), 411-417.

[4] Ghanbari, N. (2022). On the Sombor characteristic polynomial and Sombor energy of a graph. *Computational and Applied Mathematics, 41*(242).

[5] Rather, B. A., & Imran, M. (2022). Sharp bounds on the Sombor energy of graphs. *MATCH Communications in Mathematical and in Computer Chemistry, 88*(3), 605-624.

[6] Tabassum, H., Kaemawichanurat, P., Wiroonsri, N., & Adeela, E. (2023). Relationship between ordinary, Laplacian, Randić, incidence, and Sombor energies of trees. *MATCH Communications in Mathematical and in Computer Chemistry, 90*(3), 743-763.

[7] Romdhini, M. U., & Nawawi, A. (2024). On the spectral radius and Sombor energy of the non-commuting graph for dihedral groups. *Malaysian Journal of Fundamental and Applied Sciences*, *20*(1), 65-73.

[8] Beck, I. (1988). Coloring of commutative rings. *Journal of Algebra, 116*(1), 208-226. doi:10.1016/0021-8693(88)90202-5

[9] Anderson, D. F., & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *Journal of Algebra, 217*(1), 434-447. doi:10.1006/jabr.1998.7840

[10] Mönius, K. (2021). Eigenvalues of zero-divisor graphs of finite commutative rings. *Journal of Algebraic Combinatorics, 54*(3), 787-802.

[11] Rather, B. A., Aijaz, M., Ali, F., Mlaiki, N., & Ullah, A. (2022). On distance signless Laplacian eigenvalues of zero divisor graph of commutative rings. AIMS Mathematics, 7(7), 12635–12649.

[12] Zaid, N., Sarmin, N. H., & Khasraw, S. M. S. (2024, May). On the vertices of the zero divisor graph of some finite ring of matrices of dimension two. Conference Paper.
[13] Fraleigh, J. B. (2003). *A First Course in Abstract Algebra*. Pearson Education India.
[14] Saoub, K. R. (2021). Graph theory: An introduction to proofs, algorithms, and applications. CRC Press.