Proceedings of Science and Mathematics
Faculty of science, universiti teknologi malaysia
https://science.utm.my/procscimath/
Volume 26 (2024) 65-70

The Sombor Index and Its Generalization of The Coprime Graph for the Generalized Quaternion Group

Ayes Malona Siboro[1], Fariz Maulana[2] , Na'imah Hijriati[3] , I Gede Adhitya Wisnu Wardhana[4*]
[1,2,4] Department Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Mataram, Mataram, Indonesia
[3] Department Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lambung Mangkurat, Banjarbaru, Indonesia
*Corresponding author: adhitya.wardhana@unram.ac.id

Abstract

A graph is a versatile tool used to represent real-world problems such as computer networks, the internet, telephone systems, pipelines, gas, and water distribution networks. It is also used to represent discrete objects like groups and rings. One example of group representation in a graph is the coprime graph. This study discusses the Sombor Index $SO(G)$, the Reduced Sombor Index $SO_{red}(G)$, and the Average Sombor Index $SO_{avg}(G)$ of the Coprime Graph of the Generalized Quaternion Group. The results of this study provide the general formula for the Sombor Index, the reduced Sombor index and the average Sombor index of the coprime graph for the generalized quaternion group Q_{4n} with $n = 2^k$.

Keywords: *Sombor Index, Reduced Sombor Index, Average Sombor Index. Coprime Graph*

Introduction

Graphs are a tool used to present structures in the context of the Chemical Topological Graph (CTG), which is a mathematical structure that functions to represent chemical molecules. These graphs consist of vertices representing the atoms in the molecule and edges representing the chemical bonds between these atoms. In the application of CTG, graphs are used to calculate topological indices, which are useful in predicting molecular properties [1]. These graphs enable structural analysis of molecules and aid in developing more accurate models to predict their properties [2]. Graphs are also used to represent discrete objects such as groups and rings. Some properties of graphs include vertex degree and vertex distance. The degree of a vertex v is defined as the number of vertices in the graph that are adjacent, with v, and the distance between two different vertices u and v is the length of the shortest path connecting the two vertices.

Coprime graphs of quaternion groups have been an interesting research topic in mathematics and computer science. These graphs possess unique properties related to prime numbers, and studying them helps deepen our understanding of mathematical structures. In this study, we will discuss the topological indices of coprime graphs of quaternion groups [3]. Topological indices are metrics used to measure graph characteristics such as distance, density, and other special properties. Research on graph representation in quaternion groups is a broad topic with applications in various fields such as chemistry, biology, and information technology [4]. A quaternion graph represents the structure of a quaternion group, which consists of multiple quaternions with specific operations. A quaternion is a complex number with four components, and a quaternion group is a collection of different quaternions with defined operations.

Topological indices are the subject of extensive research as effective numerical descriptors for describing the shape of graphs. Topological indices can measure various graph properties, such as density, distance, and internal structure. Based on the explanation in the previous paragraph regarding the coprime graph of quaternion group, Nurhabibah and colleagues have obtained the coprime graph form from the quaternion group [5]. Therefore, we continue our research by focusing on the topological indices of coprime graphs of quaternion groups and how they can help in understanding deeper mathematical structures as explained by Theorem 2. To explore further, we have read several studies that discuss topological indices in various graphs, particularly in the coprime graphs of quaternion groups. We observe that this specific case has not been previously researched in the context of this article.

By examining the topological indices of coprime graphs of quaternion groups, we aim to uncover deeper mathematical structures for the topological indices of the Generalized Quaternion group with specific order.

Materials and methods

In this research, we used the literature study method. We looks for references related to general quaternion groups and their graphs, as well as theorems related to the Sombor index. Literature comes from national journals, theses and other references. We then reads and understands the literature obtained, noting important points such as definitions, theorems, equations, and other relevant information. We analyzes elements of each order of n values, including graph shape, vertex degrees in existing graphs, and calculates the Sombor index for several n values. We also analyzes the general form of the graph obtained. Based on these stages, the general form for each Sombor index is determined using proof. Finally, we draws up conclusions from the research results.

Results and discussion

This research discusses the Sombor Index, the Reduced Sombor Index, and the Average Sombor Index of coprime graphs of the generalized quaternion group. Below are definitions of the generalized quaternion group and the coprime graph.

Definition 1 [5]

The generalized quaternion group (Q_{4n}) with $n \geq 2$ is a group of order $4n$ formed by elements a, b which are denoted by

$$\langle a, b \mid a^{2n} = e, b^4 = e, bab^{-1} = a^{-1} \rangle.$$

Definition 2 [8]

Let G be a finite group. The coprime graph of the group G, denoted by Γ_G, is a graph with vertices consisting of all elements G, where two distinct vertices x and y are adjacent if and only if $gcd(|x|, |y|) = 1$.

Definition 3 [4]

The degree of a graph is the number of edges that are incident to the vertex. It is denotated as $deg(a)$ for a any vertex.

In this research, the general form of each type of Sombor index will be determined, among others the average of the Sombor index and the reduced Sombor index of the generalized group number four.In this research, the discussion focuses on the sombor index, avarage sombor index and reduced sombor index on coprime graphs of the generalized quaternion group with $n = 2^k$.

Theorem 1 [9]

Let Q_{4n} be a generalized quaternion group. If $n = 2^k$ then the coprime graph of Q_{4n} is a complete bipartite graph.

It is worth noting that the complete bipartite graph mentioned in Theorem 1 above is indeed the star graph $K_{1,4n-1}$[9].

Definition 3 [10]

Let G be a simple connected graph with $V(G)$ being the set of vertices, and $E(G)$ being the set of edges. The Sombor index ($SO(G)$), the reduced Sombor index $SO_{red}(G)$, the average Sombor index ($SO_{avg}(G)$) of graph G are defined as follows:

$$SO(G) = \sum_{\{uv\}\in E(G)} \sqrt{\deg(u)^2 + \deg(v)^2}$$

$$SO_{red}(G) = \sum_{\{uv\}\in E(G)} \sqrt{(\deg(u)-1)^2 + (\deg(v)-1)^2}$$

$$SO_{avg}(G) = \sum_{\{uv\}\in E(G)} \sqrt{\left(\deg(u) - \frac{2m}{n}\right)^2 + \left(\deg(v) - \frac{2m}{n}\right)^2}$$

where $deg(u)$ is the degree of vertex u, and m is the number of edges and n the number of vertices.

Example 1

Suppose $n = 2^1$, then we have $\Gamma_{Q_{4.2}} = \{a, a^2, a^3, e, b, ab, a^2b, a^3b\}$

Table 1: Neighborhood table in $\Gamma_{Q_{4.2}}$.

Elements	a	a^2	a^3	e	b	ab	a^2b	a^3b
Orders	4	2	4	1	4	4	4	4

Based on Theorem 1, the graph $\Gamma_{Q_{4n}}$ is a complete bipartite graph, more specifically a star graph. As a result, the degree of vertex e is 7 which is obtained with the generalization $4n-1$ and the degrees of the other vertices are 1. So, the sombor $SO(G)$ index from $\Gamma_{Q_{4n}}$, is calculated as follows.

$$\begin{aligned} SO(\Gamma_{Q_8}) &= \sum_{\{uv\}\in E(G)} \sqrt{\deg(u)^2 + \deg(v)^2} \\ &= \sqrt{\deg(e)^2 + \deg(a)^2} + \sqrt{\deg(e)^2 + \deg(a^2)^2} \\ &\quad + \sqrt{\deg(e)^2 + \deg(a^3)^2} + \sqrt{\deg(e)^2 + \deg(b)^2} \\ &\quad + \sqrt{\deg(e)^2 + \deg(ab)^2} + \sqrt{\deg(e)^2 + \deg(a^2b)^2} \\ &\quad + \sqrt{\deg(e)^2 + \deg(a^3b)^2} \\ &= (4n-1)\sqrt{(7)^2 + (1)^2} \\ &= (7)\sqrt{(7)^2 + (1)^2} \\ &= (7)(5)\sqrt{2} \\ SO(\Gamma_{Q_8}) &= 35\sqrt{2}. \end{aligned}$$

The Sombor index of the coprime graph of the generalized quaternion group $SO(\Gamma_{Q_8}) = 35\sqrt{2}$

The reduced Sombor index $SO_{red}(G)$ from $\Gamma_{Q_{4n}}$, is calculated as follows.

$$\begin{aligned} SO_{red}(\Gamma_{Q_8}) &= \sum_{\{u,v\}\in E(\Gamma_{\mathbb{Z}_n})} \sqrt{(\deg(u)-1)^2 + (\deg(v)-1)^2} \\ &= (4n-1)\sqrt{(7-1)^2 + (1-1)^2} \\ &= (7)\sqrt{(6)^2 + 0} \\ &= 7\sqrt{36} \end{aligned}$$

$$SO_{red}(\Gamma_{Q_8}) = 42.$$

The reduced Sombor index of the coprime graph of the generalized quaternion group $SO_{red}(\Gamma_{Q_8}) = 42$

$$SO_{avg}(\Gamma_{Q_8}) = \sum_{\{u,v\}\in E(\Gamma_{\mathbb{Z}_n})} \sqrt{\left(\deg(u) - \frac{2m}{n}\right)^2 + \left(\deg(v) - \frac{2m}{n}\right)^2}$$

$$= (4n-1)\sqrt{\left((4n-1) - \frac{2(4n-1)}{4n}\right)^2 + \left(1 - \frac{2(4n-1)}{4n}\right)^2}$$

$$= 7\sqrt{\left(\frac{n(4n-1)-2(4n-1)}{4n}\right)^2 + \left(\frac{n-2(4n-1)}{4n}\right)^2}$$

$$= 7\sqrt{\left(\frac{7-14}{8}\right)^2 + \left(\frac{-13}{8}\right)^2}$$

$$SO_{avg}(\Gamma_{Q_8}) = 7\sqrt{27{,}25}.$$

The average Sombor index of the coprime graph of the generalized quaternion group $SO_{avg}(\Gamma_{Q_8})$=$7\sqrt{27{,}25}$

Based on the examples above, the calculation of the Sombor index and its generalization can be formulated into general formulas in the following theorems.

Theorem 2 *Let Q_{4n} be a group of integers modulo n, with $n = 2^k$ for prime numbers p and natural number k. Then, the Sombor index given by* $SO(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{16n^2 - 8n + 2}$

Proof :

Since the graph is a star graph with center e, its degree is $deg\ (e) = 4n$ –1 and $deg\ (x) = 1$ for all $x \in V(G), x \neq e$

$$SO(\Gamma_{Q_{4n}}) = \sum_{\{u,v\}\in E(\Gamma_{\mathbb{Z}_n})} \sqrt{\deg(u)^2 + \deg(v)^2}$$

$$= \sqrt{(4n-1)^2 + (1)^2} + \cdots + \sqrt{(4n-1)^2 + (1)^2}$$

$$= (4n-1)\sqrt{((4n-1)\hat{}2 + (1)\hat{}2)}$$

$$= (4n-1)\sqrt{16n^2 - 8n + 1 + 1}$$

$$SO(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{16n^2 - 8n + 2} \qquad \blacksquare$$

Theorem 3 *Let Q_{4n} be a group of integers modulo n, with $n = 2^k$ for prime numbers p and natural number k. Then, the reduced Sombor index given by* $SO_{red}(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{16n^2 - 16n + 4}$

Proof:

Since the graph is a star graph with a center e, its degree is $deg\ (e) = 4n - 1$ and $deg\ (x) = 1\ for\ all\ x \in V(G), x \neq e$,

$$SO_{red}(\Gamma_{Q_{4n}}) = \sum_{\{u,v\}\in E(\Gamma_{\mathbb{Z}_n})} \sqrt{(\deg(u)-1)^2 + (\deg(v)-1)^2}$$

$$= (4n-1)\sqrt{((4n-1)-1)^2 + (1-1)^2}$$

$$= (4n-1)\sqrt{(4n-2)^2 + 0}$$

$$SO_{red}(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{16n^2 - 16n + 4} \qquad \blacksquare$$

Theorem 4 *Let Q_{4n} be a group of integers modulo n, with $n = 2^k$ for prime numbers p and natural number k. Then, the average Sombor index is given by* $SO_{avg}(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{(\frac{32^2-24n+5}{16n^2})}$

Proof:

Since the graph is a star graph with a center e, its degree is $deg\ (e) = 4n-1$ and d$eg\ (x) = 1$ for all $x \in V(G), x \neq e$

$$SO_{avg}(\Gamma_{Q_{4n}}) = \sum_{\{u,v\}\in E(\Gamma_{\mathbb{Z}_n})} \sqrt{\left(\deg(u) - \frac{2m}{n}\right)^2 + \left(\deg(v) - \frac{2m}{n}\right)^2}$$

$$= (4n-1)\sqrt{\left((4n-1) - \frac{2(4n-1)}{4n}\right)^2 + \left(1 - \frac{2(4n-1)}{4n}\right)^2}$$

$$= (4n-1)\sqrt{\left(\frac{4n-1-8n+2}{4n}\right)^2 + \left(\frac{4n-8n+2}{4n}\right)^2}$$

$$= (4n-1)\sqrt{\left(\frac{-4n+1}{4n}\right)^2 + \left(\frac{-4n+2}{4n}\right)^2}$$

$$= (4n-1)\sqrt{\left(\frac{16^2-8n+1}{16n^2}\right) + \left(\frac{16^2-16n+4}{16n^2}\right)}$$

$$SO_{avg}(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{\left(\frac{32^2-24n+5}{16n^2}\right)} \quad \blacksquare$$

Conclusion

Based on the above discussion, the general formula for the Sombor index and its generalization for the generalized quaternions groups with $n = 2^k$ has been obtained as follows. The Sombor index is given by $SO(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{16n^2 - 8n + 2}$, the reduced Sombor index given by $SO_{red}(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{16n^2 - 16n + 4}$, and the average Sombor index given by $SO_{avg}(\Gamma_{Q_{4n}}) = (4n-1)\sqrt{(\frac{32^2-24n+5}{16n^2})}$.

Acknowledgement

We would like to express our sincere thanks to our supervisors and the Algebra and Combinatorics Research Group at the University of Mataram for their valuable guidance, support and financial assistance, which played an important role in the completion of this article.

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