Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Volume 24 (2024) 65-70

The Laplacian Energy of The Conjugacy Class Graphs Associated to Generalized Quaternion Groups Up to Order 16

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Abstract
The energy of a graph was first defined in 1978. It is used in chemistry to approximate the total π−electron energy of molecules. In 2006, Laplacian energy of a graph is introduced which represents the summation of the absolute values of the Laplacian eigenvalues of the graph. In this paper, the scope of the graphs associated to groups is conjugacy class graph. A conjugacy class graph of a finite group is a graph whose vertices are non-central conjugacy classes of the group, and two vertices are connected by an edge if and only if their cardinalities are not coprime. The main objective of this research is to determine the Laplacian energy of the conjugacy class graph associated to generalized quaternion groups up to order 16. The computation of the Laplacian energy is done by using the definitions of Laplacian energy, conjugacy class graph and generalized quaternion group.
Keywords: Laplacian energy, conjugacy class graph, graph theory, group theory

Introduction

The Laplacian energy of a graph is a spectral measure of the graph, defined as the sum of the absolute values of the difference between the Laplacian eigenvalues and the ratio of twice the number of edges divided by the number of vertices. It is a relatively new concept, having been introduced by Gutman and Zhou [1]. However, it has quickly gained popularity due to its potential applications in other field such as chemistry, physics and computing.

There has been some work done in this area in recent years. For example, Das et al. [2] studied the energy and Laplacian energy of chain graphs. Meanwhile, D'Souza et al. [3] emphasized on the Laplacian energy of partial complement of a graph.

The conjugacy class graph has been introduced by Bertram et al. in 1990 [4] and many researchers are interested in studying the relation of conjugacy class graph with groups. Bhowal et al. [5] published an article about conjugacy class graph, focusing on finite groups, while Mohd Noor et al. [6] studied the conjugacy class graphs of some three-generator groups.

Roslly et al. [7] studied on generalization of randić index of the noncommuting graph for some finite groups including the generalized quaternion group. In 2023, Ismail et al. also computed several Zagreb indices of power graphs of finite non-abelian groups including the generalized quaternion group [8].

The main purpose of this study is to determine the Laplacian energy of the conjugacy class graph associated to generalized quaternion groups up to order 16, namely of order 8, 12 and 16. First, the conjugacy classes of the generalized quaternion group up to order 16 are obtained using the Cayley table of the generalized quaternion groups and the definition of conjugacy classes. Next, by the definition of the conjugacy class graph, the graphs are constructed. Finally, the Laplacian energy of each graph is calculated using its definition.

This paper is structured as follows: in Section 1, the previous studies for the topics are introduced while in Section 2, the preliminary results that are being used for this study are shown. In Section 3, the main results are presented and proved in the form of propositions. Finally, Section 4 gives the conclusion to the main results.

Preliminaries
The following are some definitions that are needed and used in this research.

Definition 1 [9] Generalized Quaternion Group
The generalized quaternion group, denoted by Q_{4n}, is a non-abelian group of order $4n$, where n is an integer and $n \geq 2$, with the group presentation:

$$Q_{4n} = \langle a, b | a^n = b^2, a^{2n} = e = b^4, b^{-1}ab = a^{-1} \rangle,$$

and the center of Q_{4n} is $Z(Q_{4n}) = \{e, a^n\}$.

Definition 2 [10] Conjugacy Classes
Let g and b be elements of a group G. Then, g and b are conjugates in G if $xgx^{-1} = b$ for some x in G. The conjugacy class of g is the set $cl(g) = \{xgx^{-1} | x \in G\}$, where the non-central conjugacy class is the conjugacy class which does not include the center elements of the group.

Definition 3 [11] Complete Graph
Let Γ be a simple graph. Then Γ is called a complete graph with n vertices and is denoted by K_n if there is an edge between any two arbitrary vertices.

Definition 4 [4] Conjugacy Class Graph
Let G be a finite group. The conjugacy class graph, denoted as Γ_G^{cl}, is the graph with the set of vertices $V(\Gamma_G^{cl}) = \{v_1, v_2, \ldots, v_n\}$ represented by the non-central conjugacy classes of G. Two vertices v_1 and v_2 are adjacent if their cardinalities are not coprime or $gcd(|v_1|, |v_2|) \neq 1$.

Definition 5 [12] Laplacian Matrix
Let Γ be a graph with the vertex-set $V(\Gamma) = \{1, \ldots, n\}$ and the edge-set $E(\Gamma) = \{e_1, \ldots, e_m\}$. The Laplacian matrix of Γ, denoted by $L(\Gamma)$ is an $n \times n$ matrix defined as follows: the rows and the columns of $L(\Gamma)$ are indexed by $V(\Gamma)$. If $i \neq j$, then a_{ij} is 0 if vertex i and j are not adjacent, and it is -1 if i and j are adjacent. The a_{ii} entry of $L(\Gamma)$ is d_i, the degree of vertex i, $i = 1,2,3, \ldots, n$.

Definition 6 [13] Characteristic Polynomial
Let A be a $n \times n$ matrix. The determinant, $\det(A - \lambda I)$ is a polynomial in the (complex) variable λ of degree n and is called the characteristic polynomial of A. The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A.

Definition 7 [13] Eigenvalues of Matrix
The roots of the characteristic equation $\det(A - \lambda I) = 0$ of A are called the eigenvalues of A.

Definition 8 [1] Laplacian Energy
Let Γ be a simple graph, L be its Laplacian matrix and $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian matrix. Then, the Laplacian energy of the graph Γ, denoted by $LE(\Gamma)$, is $LE(\Gamma) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$, where n is the number of vertices of the graph Γ and m is the number of its edges.

Main results
This section is divided into three subsections. In the first section, the conjugacy classes of the generalized quaternion group of order 8, 12 and 16 are obtained. In the second section, the conjugacy class graphs are constructed. Lastly, the Laplacian energy of the conjugacy class graphs is computed in the third section.

3.1 The Conjugacy Classes of Generalized Quaternion Groups Up to Order 16

In this section, the conjugacy classes of generalized quaternion groups of order 8, 12 and 16 are found using its definition and the Cayley table of the specified generalized quaternion groups.

Proposition 1 Let Q_8 be the generalized quaternion group of order 8, where $Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Then, the conjugacy classes of Q_8 are $cl(e) = \{e\}$, $cl(a) = \{a, a^3\}$, $cl(a^2) = \{a^2\}$, $cl(b) = \{b, a^2b\}$ and $cl(ab) = \{ab, a^3b\}$.

Proof Let Q_8 be the generalized quaternion group of order 8. By Definition 1 with $n = 2$, $Q_8 = \langle a, b | a^2 = b^2, a^4 = e = b^4, b^{-1}ab = a^{-1} \rangle = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Using Definition 2 and Cayley table of Q_8,

$cl(e) = \{xex^{-1} | x \in Q_8\} = \{e\}$,
$cl(a) = \{xax^{-1} | x \in Q_8\} = \{a, a^3\} = cl(a^3)$,
$cl(a^2) = \{xa^2x^{-1} | x \in Q_8\} = \{a^2\}$,
$cl(b) = \{xbx^{-1} | x \in Q_8\} = \{b, a^2b\} = cl(a^2b)$ and
$cl(ab) = \{x(ab)x^{-1} | x \in Q_8\} = \{ab, a^3b\} = cl(a^3b)$. □

Proposition 2 Let Q_{12} be the generalized quaternion group of order 12, where $Q_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. Then, the conjugacy classes of Q_{12} are $cl(e) = \{e\}$, $cl(a) = \{a, a^5\}$, $cl(a^2) = \{a^2, a^4\}$, $cl(a^3) = \{a^3\}$, $cl(b) = \{b, a^2b, a^4b\}$ and $cl(ab) = \{ab, a^3b, a^5b\}$.

Proof The proof is similar to the proof in Proposition 1. □

Proposition 3 Let Q_{16} be the generalized quaternion group of order 16, where $Q_{16} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$. Then, the conjugacy classes of Q_{16} are $cl(e) = \{e\}$, $cl(a) = \{a, a^7\}$, $cl(a^2) = \{a^2, a^6\}$, $cl(a^3) = \{a^3, a^5\}$, $cl(a^4) = \{a^4\}$, $cl(b) = \{b, a^2b, a^4b, a^6b\}$ and $cl(ab) = \{ab, a^3b, a^5b, a^7b\}$.

Proof The proof is similar to the proof in the previous two propositions. □

3.2 The Conjugacy Class Graphs of Generalized Quaternion Groups Up to Order 16

In this section, the conjugacy class graphs of generalized quaternion groups of order 8, 12 and 16 are constructed using its definition.

Proposition 4 Let Q_8 be the generalized quaternion group of order 8, where $Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Then, the conjugacy class graph of Q_8, $\Gamma_{Q_8}^{cl} = K_3$.

Proof By using Definition 4, the vertices of the conjugacy class graph are its non-central conjugacy classes. Thus, by Proposition 1, $V(\Gamma_{Q_8}^{cl}) = \{cl(a), cl(b), cl(ab)\}$, since both $cl(e)$ and $cl(a^2)$ are elements of the center of Q_8, they will not be considered as vertices for the graph. By Definition 4, two vertices v_1 and v_2 are adjacent if $gcd(|v_1|, |v_2|) \neq 1$. By Proposition 1, $|cl(a)| = |cl(b)| = |cl(ab)| = 2$, thus, each pair has a greatest common divisor equal to 2. Hence, the set of edges is $E(\Gamma_{Q_8}^{cl}) = \{\{cl(a), cl(b)\}, \{cl(a), cl(ab)\}, \{cl(b), cl(ab)\}\}$. Thus, by Definition 3, the conjugacy class graph of Q_8 is a complete graph with 3 vertices, as shown in Figure 1.

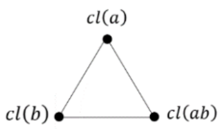

Figure 1 The conjugacy class graph of Q_8, $\Gamma_{Q_8}^{cl}$. □

Proposition 5 Let Q_{12} be the generalized quaternion group of order 12, where $Q_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. Then, the conjugacy class graph of Q_{12}, $\Gamma_{Q_{12}}^{cl} = 2K_2$.

Proof The proof is similar to the proof in Proposition 4. Figure 2 shows the conjugacy class graph of Q_{12}.

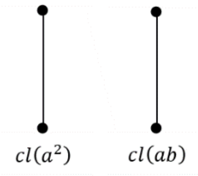

Figure 2 The conjugacy class graph of Q_{12}, $\Gamma_{Q_{12}}^{cl}$. □

Proposition 6 Let Q_{16} be the generalized quaternion group of order 16, where $Q_{16} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$. Then, the conjugacy class graph of Q_{16}, $\Gamma_{Q_{16}}^{cl} = K_5$.

Proof The proof is similar to the proof in the previous two propositions. Figure 3 shows the conjugacy class graph of Q_{16}.

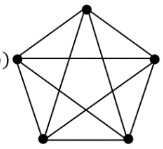

Figure 3 The conjugacy class graph of Q_{16}, $\Gamma_{Q_{16}}^{cl}$. □

3.3 The Laplacian Energy of The Conjugacy Class Graphs of Generalized Quaternion Groups Up to Order 16

In this section, the Laplacian energy of the conjugacy class graphs of generalized quaternion groups of order 8, 12 and 16 are computed using its definition.

Proposition 7 Let Q_8 be the generalized quaternion group of order 8 and $\Gamma_{Q_8}^{cl}$ be its conjugacy class graph. Then, the Laplacian energy of $\Gamma_{Q_8}^{cl}$, $LE(\Gamma_{Q_8}^{cl}) = 4$.

Proof Based on Definition 5, the rows and columns of the Laplacian matrix of $\Gamma_{Q_8}^{cl}$ can be indexed by $V(\Gamma_{Q_8}^{cl})$. Thus, the Laplacian matrix of $\Gamma_{Q_8}^{cl}$ is obtained as follows:

$$L(\Gamma_{Q_8}^{cl}) = \begin{array}{c} \\ cl(a) \\ cl(b) \\ cl(ab) \end{array} \begin{array}{c} \begin{array}{ccc} cl(a) & cl(b) & cl(ab) \end{array} \\ \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{array}.$$

Then, by Definition 6, the characteristic polynomial of $L(\Gamma_{Q_8}^{cl})$,

$$\left|L(\Gamma_{Q_8}^{cl}) - \lambda I\right| = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = \lambda(\lambda-3)^2.$$

Using Definition 7, the eigenvalues are 0 and 3 (with multiplicities 2) where $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 3$. Thus, by Definition 8, the Laplacian energy of $\Gamma^{cl}_{Q_8}$,

$$LE(\Gamma^{cl}_{Q_8}) = \sum_{i=1}^{3}\left|\mu_i - \frac{2(3)}{3}\right| = \left|0 - \frac{2(3)}{3}\right| + \left|3 - \frac{2(3)}{3}\right| + \left|3 - \frac{2(3)}{3}\right| = 4.$$ □

Proposition 8 Let Q_{12} be the generalized quaternion group of order 12 and $\Gamma^{cl}_{Q_{12}}$ be its conjugacy class graph. Then, the Laplacian energy of $\Gamma^{cl}_{Q_{12}}$, $LE(\Gamma^{cl}_{Q_{12}}) = 4$.

Proof The Laplacian matrix of $\Gamma^{cl}_{Q_{12}}$ is obtained as follows:

$$L(\Gamma^{cl}_{Q_{12}}) = \begin{array}{c} \\ cl(a) \\ cl(a^2) \\ cl(b) \\ cl(ab) \end{array}\begin{array}{c} \begin{array}{cccc} cl(a) & cl(a^2) & cl(b) & cl(ab) \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}.$$

The characteristic polynomial of $L(\Gamma^{cl}_{Q_{12}})$,

$$\left|L(\Gamma^{cl}_{Q_{12}}) - \lambda I\right| = \begin{vmatrix} 1-\lambda & -1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & -1 & 1-\lambda \end{vmatrix} = \lambda^2(\lambda-2)^2.$$

The eigenvalues are 0 (with multiplicities 2) and 2 (with multiplicities 2) where $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 2,\ \lambda_4 = 2$. Thus, the Laplacian energy of $\Gamma^{cl}_{Q_{12}}$,

$$LE(\Gamma^{cl}_{Q_{12}}) = \left|0 - \frac{2(2)}{4}\right| + \left|0 - \frac{2(2)}{4}\right| + \left|2 - \frac{2(2)}{4}\right| + \left|2 - \frac{2(2)}{4}\right| = 4.$$ □

Proposition 9 Let Q_{16} be the generalized quaternion group of order 16 and $\Gamma^{cl}_{Q_{16}}$ be its conjugacy class graph. Then, the Laplacian energy of $\Gamma^{cl}_{Q_{16}}$, $LE(\Gamma^{cl}_{Q_{16}}) = 8$.

Proof The Laplacian matrix of $\Gamma^{cl}_{Q_{16}}$ is obtained as follows:

$$L(\Gamma^{cl}_{Q_{16}}) = \begin{array}{c} \\ cl(a) \\ cl(a^2) \\ cl(a^3) \\ cl(b) \\ cl(ab) \end{array}\begin{array}{c} \begin{array}{ccccc} cl(a) & cl(a^2) & cl(a^3) & cl(b) & cl(ab) \end{array} \\ \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \end{array}.$$

The characteristic polynomial of $L(\Gamma^{cl}_{Q_{16}})$,

$$\left|L(\Gamma^{cl}_{Q_{16}}) - \lambda I\right| = \begin{vmatrix} 4-\lambda & -1 & -1 & -1 & -1 \\ -1 & 4-\lambda & -1 & -1 & -1 \\ -1 & -1 & 4-\lambda & -1 & -1 \\ -1 & -1 & -1 & 4-\lambda & -1 \\ -1 & -1 & -1 & -1 & 4-\lambda \end{vmatrix} = \lambda(\lambda-5)^4.$$

The eigenvalues are 0 (with multiplicities 2) and 2 (with multiplicities 2) where $\lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 5, \lambda_4 = 5, \lambda_5 = 5$. Thus, the Laplacian energy of $\Gamma^{cl}_{Q_{16}}$,

$$LE(\Gamma^{cl}_{Q_{16}}) = \left|0 - \frac{2(10)}{5}\right| + \left|5 - \frac{2(10)}{5}\right| + \left|5 - \frac{2(10)}{5}\right| + \left|5 - \frac{2(10)}{5}\right| + \left|5 - \frac{2(10)}{5}\right| = 8.$$ □

Conclusion

In conclusion, all the conjugacy class graphs associated to generalized quaternion group up to order 16 found are either a complete graph or a union of complete graphs. Meanwhile, their Laplacian energy turned out to be all even.

Acknowledgement

The authors would like to express their sincere gratitude to all who have contributed throughout the course of this work.

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