Proceedings of Science and Mathematics
Faculty of science, universiti teknologi malaysia
https://science.utm.my/procscimath/
Volume 26 (2024) 79-84

Topological Indices of GCD Graph Representations for Integer Modulo Groups with Prime Power Order

Rendi Bahtiar Pratama[a], Fariz Maulana[b], Abdurahim[c*], I Gede Adhitya Wisnu Wardhana[d]
[a,b,c,d]Department Mathematics, Faculty of Mathematics and Natural Science, Universitas Mataram, Indonesia
*Corresponding author: abdurahim@staff.unram.ac.id

Abstract
Graph is a concept in mathematics used to visualize the relationship between one object and another. In mathematics the group is an abstract object, so with graph we can visualize the group, by analyzing the relationship between each member in the group. In this study, we used GCD graph to represent the integers modulo group. The GCD graph on the group is defined as follows: two distinct vertices x and y in G are adjacent if and only if $(|x - y|, n) = d$ where d is a member of the set of factors of n. The author presents the results of this study, which include the structure of the group of integers using the GCD graph and several topological indices such as polynom sombor, ABC, and forgotten index. Whenever the order of the group is prime power.
Keywords: GCD Graph; Integer Group; Topological Indices

Introduction
In mathematics, graph theory is an interesting branch to explore, because studying and analyzing the resulting graph shapes can help solve various problems. Graphs are used to visualize abstract objects, such as group [6]. As for the definition of a group, a group is a non-empty set that satisfies the closed, associative, identity and inverse properties. In this study the group used is the group of integers modulo n which is denoted by $\mathbb{Z}_n$ [11]. There are many examples of graph representations on groups that have been researched, such as power graph representations on the group of integers modulo n and the dihedral group [3]. The GCD representation of a graph on the group of integers modulo n is defined as a graph whose vertices are all members of the group, where two different members x and y of the group are neighboring if and only if $(|x - y|, n) = d$, where n is a prime power, is the graph defined only for $n = p^k$ and d is a member of the set of all factors of n denoted by $D_n = \{p^0, p^1, p^2, \dots, p^k\}$ [10].

In chemistry, the graph representation of the group can be used to determine several topological indexes based on degree of the vertices, which can be utilized to assist in understanding the chemical bonding and electronic properties of a compound, the physical characteristics of chemical compounds, chemical reactivity, and biological activity. Currently there are several indexes being studied, namely polynom, Atom-Bond-Conectivity (ABC), and Forgotten [1] [4] [5]. The connection between graphs and molecular structure in chemistry is a clear example of how mathematical concepts can be applied to solve real problems in science.

Materials and methods
Is discussed in this reseach the representation of the GCD graph on the group. The following are some definitions and theorems used:

Definitation 1 [10] GCD Graph in group G is defined as a graph whose vertices are the set of all members in group G denoted by Γ_G, and two vertices x and y are neighbors if and only if $(|x - y|, n) = d$, where d is factor of n.

The d is one of all factors of n, to make it easier to write it down, we define the set of all factors of n.

Definitation 2 [10] GCD graph on group G $(\mathbb{Z}_n)$. the set of all factors of n, where n is a prime power $(n = p^k)$, notated as $D_{p^k} = \{p^0, p^1, p^2, \dots, p^{k-1}, p^k\}$.

Integers modulo n is the set of integers divided into multiple groups based on the remainder of the quotient when divided by n. This set is often called $(\mathbb{Z}_n)$.

Defination 3 [8] Group G is called integer group modulo n, with its members being positive integers with addition operation modulo n. Denoted by $\mathbb{Z}_n = \{0,1,2, \dots, n-1\}$.

Definition 4 [2] Suppose Γ_G is a graph where $V(\Gamma_G)$ is a set of vertex $a \in V(\Gamma_G)$. The number of edges connecting neighbor a with the other vertex, denoted by $\deg(a)$.

In this research we will look at several topological indices, including Polynom sombor indices, Atom-Bond-Conectivity (ABC) index, and Forgoten index. The following is the definition.

Definition 5 [3] Given Γ_G a simple connected graph with $V(\Gamma_G)$ is the set of vertices, and $E(\Gamma_G)$ is the set of edges, the Sombor Polynom is defined

$$So(\Gamma_G; x) = \sum_{xy \in E(\Gamma_G)} \frac{1}{\sqrt{\deg(x)^2 + \deg(y)^2}} x^{\deg(x)^2 + \deg(y)^2}$$

Definition 6 [1] Given Γ_G a simple connected graph with $V(\Gamma_G)$ is the set of vertices, and $E(\Gamma_G)$ is the set of edges, the Atom-Bond-Connectivity (ABC) index is defined

$$ABC(\Gamma_G) = \sum_{xy \in E(\Gamma_G)} \sqrt{\frac{\deg(x) + \deg(y) - 2}{\deg(x)\deg(y)}}$$

Definition 7 [4] Given Γ_G a simple connected graph with $V(\Gamma_G)$ is the set of vertices, and $E(\Gamma_G)$ is the set of edges, the Forgotten index is defined

$$F(\Gamma_G) = \sum_{xy \in E(\Gamma_G)} \deg(x)^2 + \deg(y)^2$$

This research is carried out starting with several stages of research starting with a literature study where researchers search for and analyze references related to gcd graphs, groups, graph representations in groups and index topology. After getting some important points from the references used, it is used to get how the representation of the gcd graph in the group of integers modulo n with n prime power. then by knowing the form of the gcd graph in the group, the degree of each vertex is obtained which is used to find several topological indexes.

Results and discussion

To make it clearer, let's see an example of gcd graph on the group of integers modulo n, with n is a prime powers.

Example 1. In the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $D_4 = \{1, 2, 4\}$.

Table 1: Order of the element in $\mathbb{Z}_4$

Element $\mathbb{Z}_4$	**0**	**1**	**2**	**3**
Orde	1	4	2	4

To determine the neighborhood of vertices in the GCD graph of $\mathbb{Z}_4$, based on the **Definition 1** the GCD of the order of the difference of two vertices with n is calculated. In other words $(|x - y|, n) = d$.

Table 2: The neighborhood in the GCD graph of $\mathbb{Z}_4$

$\mathbb{Z}_4$	**0**	**1**	**2**	**3**
0	-	$(1,4)=1$	$(2,4)=2$	$(3,4)=1$
1	$(1,4)=1$	-	$(1,4)=1$	$(2,4)=2$
2	$(2,4)=2$	$(1,4)=1$	-	$(1,4)=1$
3	$(3,4)=1$	$(2,4)=2$	$(1,4)=1$	-

From the above neighboring table, we get the GCD of the graph in group $\mathbb{Z}_4$ as follows:

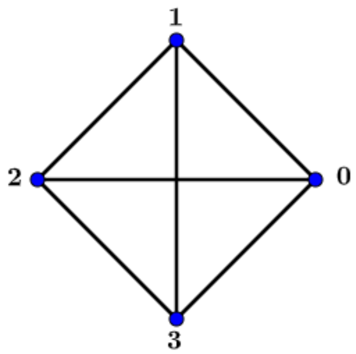

Figure 1. GCD graph in $\mathbb{Z}_4$

Find the indexes of the polynom sombor, ABC, and forggoten of $\mathbb{Z}_4$.Based on Definition 5,6,and 7 we get the formula to find the indices as follows:

Polynom sombor :

$$\begin{aligned} So(\Gamma_{\mathbb{Z}_4};x) &= \sum_{xy\,\in\,E(\Gamma_{\mathbb{Z}_4})} \frac{1}{\sqrt{\deg(x)^2+\deg(y)^2}}x^{\deg(x)^2+\deg(y)^2} \\ &= C_2^4\ \frac{1}{\sqrt{\deg(x)^2+\deg(y)^2}}x^{\deg(x)^2+\deg(y)^2} \\ &= \frac{4(4-1)}{2}\ \frac{1}{\sqrt{2(4-1)^2}}x^{2(4-1)^2} \\ &= \frac{12}{2}\ \frac{1}{\sqrt{18}}x^{18} \\ &= \frac{12}{2}\ \frac{1}{3\sqrt{2}}x^{18} \\ So(\Gamma_{\mathbb{Z}_4};x) &= \sqrt{2}x^{18} \end{aligned}$$

Atom-Bond-Connectivity (ABC):

$$\begin{aligned} ABC\ (\Gamma_{\mathbb{Z}_4}) &= \sum_{xy\,\in\,E(\Gamma_{\mathbb{Z}_4})} \sqrt{\frac{\deg(x)\ +\deg(y)-2}{\deg(x)\ \deg(y)}} \\ &= C_2^4\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)^2}} \\ &= \frac{4(4-1)}{2}\sqrt{\frac{3+1}{(3-1)^2}} \\ &= \frac{12}{2}\sqrt{\frac{4}{9}} \\ &= \frac{12}{2}\frac{2}{3} \\ ABC\ (\Gamma_{\mathbb{Z}_4}) &= 4 \end{aligned}$$

Forgotten:

$$
\begin{aligned}
F(\Gamma_{\mathbb{Z}_4}) &= \sum_{xy \in E(\Gamma_{\mathbb{Z}_4})} \deg(x)^2 + \deg(y)^2 \\
&= C_2^4 \; deg(x)^2 + deg(y)^2 \\
&= \frac{4(4-1)}{2} \; (2(4-1)^2) \\
&= \frac{12}{2} \; (2(3)^2) \\
&= 6 \; (2(3)^2) \\
F(\Gamma_{\mathbb{Z}_4}) &= 12(3)^2
\end{aligned}
$$

From the above example determining the order of n is crucial. This allows us to generalize and find the order of the group of integers modulo n, where n is a prime power. The order of n is then used to determine the form of the GCD graph of the group $\mathbb{Z}_n$, resulting in the following outcomes.

Table 3: The order in $\mathbb{Z}_n$

$\mathbb{Z}_{p^k}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{\dots}$	$\boldsymbol{p}$	$\boldsymbol{p+1}$	$\mathbf{\dots}$	$\boldsymbol{p^{k-1}}$	$\boldsymbol{p^{k-1}+1}$	$\mathbf{\dots}$	$\boldsymbol{p^{k-1}+(p-1)}$
Orde	1	p^k	$\dots$	p^l	p^k	$\dots$	p^r	p^k	$\dots$	p^k

By utilizing the generalized order table above, we determine the form of the GCD graph on the group of integers as outlined in Theorem 1 below.

Theorem 1. Given a group $\mathbb{Z}_{p^k}$ with the operation $+_{mod(p^k)}$, for $k \in \mathbb{N}$, where p prime. Then the GCD graph of the group $\mathbb{Z}_{p^k}$ is the complete graph K_{p^k}

Proof. If $D_{p^k} = \{p^0, p^1, p^2, \dots, p^{k-1}, p^k\}$, for $k \in \mathbb{N}$, with p prime. From Table 3 we find that $p||x|, \forall x \in \mathbb{Z}_{p^k}$, means $|x| = p^l, 0 \le l \le k$. Since $x - y \in \mathbb{Z}_{p^k}$, then $p^l||x-y|, |x-y| = p^r$ with $0 \le r \le k$. Thus $(|x-y|, n) = (p^r, p^k) = d \, , d \in D_{p^k}$.

By recognizing the gcd representation of the graph in the group of integers modulo, we will easily determine the degree of each vertex in the group. Based on this definition, the index value is calculated

Theorem 2. If given an integer group modulo n ($\mathbb{Z}_{p^k}$) with operation $+_{\mathrm{mod}(p^k)}$. where n is a prime power, then $SO((\Gamma_{\mathbb{Z}_n}; x)) = \frac{n\sqrt{2}}{4} x^{2(n-1)^2}$

Proof. Based on Theorem 1, we conclude that $\deg(x) = n - 1$ for all $x \in V(\Gamma)$. Since the graph $L(\Gamma; x)$ is a complete graph, all its vertices are connected. In other words, the number of vertex pairs to be summed is equal to C_2^n.

$$
\begin{aligned}
So\left(L(\Gamma_{\mathbb{Z}_n}; x)\right) &= \sum_{x,y \in E(\Gamma_{\mathbb{Z}_n})} \frac{1}{\sqrt{\deg(x)^2 + \deg(y)^2}} x^{\deg(x)^2 + \deg(y)^2} \\
&= C_2^n \; \frac{1}{\sqrt{\deg(x)^2 + \deg(y)^2}} x^{\deg(x)^2 + \deg(y)^2} \\
&= \frac{n(n-1)}{2} \; \frac{1}{\sqrt{2(n-1)^2}} x^{2(n-1)^2} \\
&= \frac{n(n-1)}{2} \cdot \frac{1}{(n-1)\sqrt{2}} x^{2(n-1)^2} \\
&= \frac{n}{2} \cdot \frac{1}{\sqrt{2}} x^{2(n-1)^2} \\
So\left(L(\Gamma_{\mathbb{Z}_n}; x)\right) &= \frac{n\sqrt{2}}{4} \; x^{2(n-1)^2}
\end{aligned}
$$

Theorem 3. If given an integer group modulo n ($\mathbb{Z}_{p^k}$) with operation $+_{\text{mod}(p^k)}$. with n is a prime power, then $ABC\ (\Gamma_{\mathbb{Z}_n}) = \frac{n}{2}\sqrt{2n-4}$

Proof. For the same reason as in the proof of Theorem 2, we conclude that

$$\begin{aligned} ABC\ (\Gamma_{\mathbb{Z}_n}) &= \sum_{xy\,\in\, E(\Gamma_{\mathbb{Z}_n})} \sqrt{\frac{\deg(x)+\deg(y)-2}{\deg(x)\deg(y)}} \\ &= C_2^n \sqrt{\frac{(n-1)+n-3}{(n-1)^2}} \\ &= \frac{n(n-1)}{2}\sqrt{\frac{2n-4}{(n-1)^2}} \\ &= \frac{n(n-1)}{2}\,\frac{\sqrt{2n-4}}{n-1} \\ ABC\ (\Gamma_{\mathbb{Z}_n}) &= \frac{n}{2}\sqrt{2n-4} \end{aligned}$$

Theorem 4. If given an integer group modulo n ($\mathbb{Z}_{p^k}$) with operation $+_{\text{mod}(p^k)}$. where n is a prime power, then $F(\Gamma_{\mathbb{Z}_n}) = n(n-1)^3$

Proof. For the same reason as in the proof of Theorem 2, we can conclude that

$$\begin{aligned} F(\Gamma_{\mathbb{Z}_n}) &= \sum_{xy\,\in\, E(\Gamma_{\mathbb{Z}_n})} \deg(x)^2+\deg(y)^2 \\ &= C_2^n\ ((n-1)^2+(n-1)^2) \\ &= \frac{n(n-1)}{2}\ (2(n-1)^2) \\ F(\Gamma_{\mathbb{Z}_n}) &= n(n-1)^3 \end{aligned}$$

Conclusion

Based on the conducted research, it has been found that the GCD representation of graphs in integer groups, with addition operation modulo n where n is a prime number, results in a complete graph. Furthermore, the research has yielded results for several topological indices: the Polynom Sombor $So\left((\Gamma_{\mathbb{Z}_n};x)\right) = \frac{n\sqrt{2}}{4}\ x^{2(n-1)^2}$, the Atom-Bond-Connectivity (ABC) $ABC\left(\Gamma_{\mathbb{Z}_n}\right) = \frac{n}{2}\ \sqrt{2n-4}$, and the Forgoten $F(\Gamma_{\mathbb{Z}_n}) = n(n-1)^3$.

Acknowledgement

We extend special acknowledgments to our supervisors and the Algebra and Combinatorics Research Group at Universitas Mataram for their invaluable guidance, support, and financial assistance, which have been instrumental in the completion of this article.

References

[1] Ali, A., Das, K. C., Dimitrov, D., & Furtula, B. (2021). Atom–bond connectivity index of graphs: A review over extremal results and bounds. In *Discrete Mathematics Letters* (Vol. 5, pp. 68–93). Shahin Digital Publisher. https://doi.org/10.47443/dml.2020.0069

[2] Asmarani, E. Y., Syarifudin, A. G., Wardhana, I. G. A. W., & Switrayni, N. W. (2022). The Power Graph of a Dihedral Group. *EIGEN MATHEMATICS JOURNAL*, 80–85. https://doi.org/10.29303/emj.v4i2.117

[3] Ghanbari, N. (2021). *On the Sombor characteristic polynomial and Sombor energy of a graph*. http://arxiv.org/abs/2108.08552

[4] Gutman, I. (2020). *Geometric Approach to Degree-Based Topological Indices: Sombor Indices*.

[5] Hadi, S., Ningrum, P., Siboro, A. M., Adhitya, G., Wardhana, W., & Awanis, Z. Y. (2024). Prosiding SAINTEK ABSTRAKSI CHEMICAL TOPOLOGICAL GRAPH (CTG) MELALUI INDEKS TOPOLOGIS GRAF ALJABAR. *LPPM Universitas Mataram*, *6*.

[6] Husni, M. N., Syafitri, H., Siboro, A. M., Syarifudin, A. G., Aini, Q., & Wardhana, I. G. A. W. (2022). THE HARMONIC INDEX AND THE GUTMAN INDEX OF COPRIME GRAPH OF INTEGER GROUP MODULO WITH ORDER OF PRIME POWER. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *16*(3), 961–966. https://doi.org/10.30598/barekengvol16iss3pp961-966

[7] Kiani, D., Aghaei, M. M. H., Meemark, Y., & Suntornpoch, B. (2011). Energy of unitary Cayley graphs and gcd-graphs. *Linear Algebra and Its Applications*, *435*(6), 1336–1343. https://doi.org/10.1016/j.laa.2011.03.015

[8] Putra, L. R. W., Awanis, Z. Y., Salwa, S., Aini, Q., & Wardhana, I. G. A. W. (2023). THE POWER GRAPH REPRESENTATION FOR INTEGER MODULO GROUP WITH POWER PRIME ORDER. *BAREKENG: Jurnal Ilmu Matematika Dan Terapan*, *17*(3), 1393–1400. https://doi.org/10.30598/barekengvol17iss3pp1393-1400

[9] Redžepović, I. (2021). Chemical applicability of Sombor indices. *Journal of the Serbian Chemical Society*, *86*(5), 445–457. https://doi.org/10.2298/JSC201215006R

[10] Sathya, Madhumitha & Naduvath, Sudev. (2023). Graphs on groups in terms of the order of elements: A Review. Discrete Mathematics, Algorithms and Applications. 16. 10.1142/S1793830923300035.

[11] Tapanyo, W., Tongpikul, T., & Kaewpradit, S. (2022). *gcd-Pairs in* $\mathbb{Z}_n$ *and their graph representations*. http://arxiv.org/abs/2206.01847