Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 24, 2024, page 90-105

Metabelian Groups of Order at Most 60

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Abstract
A group *G* is metabelian if there exists a normal subgroup N in G such that both N and the factor group, G/N are abelian. The main objective of this research is to determine all metabelian groups of order at most 60. In this research, some basic concepts of metabelian groups are presented and the determination of metabelian groups are done based on their definition and some theorems and results given by previous researchers. Identify research gaps in the study of metabelian groups of order 33 to 60. With the scope of this research, 162 from 168 groups of order from 33 to 60 are detected as metabelian groups. The rest five groups of order 48 and one group of order 54 are not metabelian. The group presentations for groups of order 33 to 60 have been analysed to determine if they are metabelian. The Groups, Algorithms and Programming (GAP) software have been used as a tool to assist some computations in determination of the metabelian groups.
Keywords: Metabelian; Commutator subgroup; Group, Algorithms and Programming (GAP); Group of Small Orders.

1. Introduction

Metabelian groups are groups that closely resemble abelian groups, but they are not necessarily the same. While every abelian group can be considered metabelian, not all metabelian groups meet the criteria for abelian groups. This resemblance is primarily seen in the specific structure of their commutator subgroups. In Russian mathematical literature, the term metabelian group is sometimes used to refer to nilpotent groups with a nilpotency class of two. Terms like "solvable length two" and "derived length two" are used on occasion to refer to metabelian groups that have a derived length of exactly two. By emphasising that these groups are metabelian but not abelian, this description provides a more accurate definition than the common interpretation of the term metabelian. Being metabelian is derived from applying the 'meta' operator to an Abelian group. Another way to describe a metabelian group is as Abelian-by-Abelian, with 'by' indicating a group extension. The metabelian property is preserved in the direct product of metabelian groups. The current research aims to identify metabelian groups with an order 33 to 60.

SOME BASIC CONCEPTS AND PROPERTIES IN METABELIAN GROUPS

In the following, the definitions and some properties of these groups are presented:

Definition 2.1 [1] Metabelian
A group *G* is metabelian if there exists a normal subgroup *N* in *G* such that both *N* and G/N are abelian.

Definition 2.2 [1] Commutator Subgroup
Given $a, b \in G$. The commutator of a and b, denoted by $[a, b]$ is the element

$$[a, b] = a^{-1}b^{-1}ab \in G$$

The commutator subgroup or derived subgroup $[G, G] \leq G$, is defined to be smallest subgroup of *G* which contains all the commutators $[a, b]$.

The following lemma and theorems have been proved by Wisnesky in [5].

The lemma is started first to prove the theorems followed.

Lemma 2.1 [5]

Let *G* be a group and *N* a subgroup of *G*. Then $gN = N$, if and only if $g \in N$.

Theorem 2.1 [5]

Let *G* be a group and *N* a normal subgroup of *G*. Then G/N is abelian if and only if the commutator subgroup $G' = [G, G] \subseteq N$.

Theorem 2.2 [3]

If the index of *H* in *G* is 2, then *H* is a normal subgroup. In symbols, we write:

$$[G : H] = 2 \Rightarrow H \lhd G .$$

Theorem 2.3 [5]

Every abelian group is metabelian.

Theorem 2.4 [5]

If *H* is a subgroup of a metabelian group *G*, then *H* is metabelian.

The following definitions and theorems will be used in proving all metabelian groups of order at most 60

Definition 2.3 [3] Cyclic Group

If *G* is a group and $a \in G$ such that $G = a = \{a^n = e, n \in N\}$, then *G* is a cyclic group generated by *a*. In this research, the cyclic group is denoted as Z_n and the order of Z_n is *n*.

Definition 2.4 [3] Dihedral Groups of Degree *n*

For $n \in Z$, and $n \geq 3$, the dihedral group, D_n, is the set of symmetries of a regular *n*-gon. Furthermore, the order of D_n is 2*n* or equivalently $|D_n| = 2n$. The dihedral groups can be represented in a form of generators and relations as given in the following:

$$\langle x,y \mid x^2 = 1, y^n = 1, (xy)^2 = 1 \rangle$$

It is known that the center of a dihedral group is the identity and the set $\{1, a^n 2\}$ when *n* is odd and even, respectively.

Definition 2.5 [9] Center of a Group *G*

The center, *Z(G)* of a group *G* is the subset of elements in *G* that commute with every element of *G.*

$$Z(G) = \{a \in G \mid ax = xa, \ \forall x \in G\}$$

Definition 2.6 [6] Semi Direct Product

Let $N \lhd G$ and there is a subgroup *H* such that $G = HN$ and $H \cap N = \{1\}$. Then *G* is said to semidirect product of *N* and *H* denoted by $G = N \rtimes H$.

Definition 2.7 [6] Normal Subgroup

A subgroup *H* of a group *G* is normal if it is left and right cosets coincide, that is if $gH = Hg$ for all

$g \in G$

Definition 2.8 [6] Group Presentation

Let *A* be a set and let $\{r_i\} \subseteq F(A)$. Let *R* be the least normal subgroup of $F(A)$ containing r_i. An isomorphism φ of $F(A)/R$ of onto a group *G* is a presentation of *G.* The sets *A* and $\{r_i\}$ give a group presentation. The set *A* is the generators of the presentation and each r_i is a relator. Each $r \in R$ is a consequence of $\{r_i\}$. An equation $r_i = 1$ is a relation. A finite presentation is one in which both *A* and $\{r_i\}$ are finite sets.

Definition 2.9 [6] Derived Length

If *G* is a soluble or solvable group, the length of a shortest abelian series in *G* is called the derived length of *G*. Thus, *G* has:

i. Derived length = 0 ↔ |*G*| = 1
ii. Derived length ≤ 1 ↔ *G* abelian group.
iii. Derived length ≤ 2 ↔ *G* metabelian group.

Theorem 2.5 [6]

A group *G* is metabelian if it has derived length ≤ 2, that is it has an abelian series.

$$1 \lhd H \lhd G$$

Theorem 2.6 [3]

The center, *Z(G)* of a group G is always normal.

Theorem 2.7 [3]

A group of prime order is cyclic.

Theorem 2.8 [3]

Every cyclic group is abelian.

Theorem 2.9 [1]

The direct product of abelian groups is abelian.

Theorem 2.10 [1]

A direct product of metabelian groups is metabelian.

Theorem 2.11 [1]

Any dihedral group is metabelian.

In 2012, a research study was conducted by Abdul Rahman and Sarmin [1] with the objective of identifying all metabelian groups having orders up to 24. Following this, in 2017, Simon [2] accomplished the task of determining all metabelian groups within the order range of 25 to 32.

Expanding on the groundwork laid by these earlier research attempt, the current study is focus on the determination of metabelian groups with orders from 33 to 60. The outcomes of this research possess significant potential for providing guidance in further investigations within related fields and areas of study. Therefore, a table of all 158 groups of order from 33 to 60 with their group presentation have been listed in Table 2.1 below.

	Groups	Group Order	Group ID in GAP	Group Presentation / Structure
1	$\mathbb{Z}_{33}$	33	[33,1]	Abelian
2	$\mathbb{Z}_{34}$	34	[34,1]	Abelian
3	D_{17}	34	[34,2]	$\langle a,b \mid a^{17}=b^2=1, bab=a^{-1}\rangle$
4	$\mathbb{Z}_{35}$	35	[35,1]	Abelian
5	Dic_9	36	[36,1]	$\langle a,b \mid a^{18}=1, b^2=a^9, bab^{-1}=a^{-1}\rangle$
6	$\mathbb{Z}_{36}$	36	[36,2]	Abelian
7	$\mathbb{Z}_3 . A_4$	36	[36,3]	$\langle a,b,c,d \mid a^3=b^2=c^2=1, d^3=a, ab=ba,$ $ac=ca, ad=da, dbd^{-1}=bc=cb, dcd^{-1}=b\rangle$
8	D_{18}	36	[36,4]	$\langle a,b \mid a^{18}=b^2=1, bab=a^{-1}\rangle$
9	$\mathbb{Z}_2\times\mathbb{Z}_{18}$	36	[36,5]	$\langle a,b \mid a^2=b^{18}=1, ab=ba\rangle$
10	$\mathbb{Z}_3\times Dic_3$	36	[36,6]	$\langle a,b,c \mid a^3=b^6=1, c^2=b^3, ab=ba,$ $ac=ca, cbc^{-1}=b^{-1}\rangle$
11	$\mathbb{Z}_3\rtimes Dic_3$	36	[36,7]	$\langle a,b,c \mid a^3=b^6=1, c^2=b^3, ab=ba,$ $cac^{-1}=a^{-1}, cbc-1=b^{-1}\rangle$
12	$\mathbb{Z}_3\times \mathbb{Z}_{12}$	36	[36,8]	$\langle a,b \mid a^3=b^{12}=1, ab=ba\rangle$
13	$\mathbb{Z}_3{}^2\rtimes \mathbb{Z}_4$	36	[36,9]	$\langle a,b,c \mid a^3=b^3=c^4=1,$ $cbc^{-1}=ab=ba, cac^{-1}=a^{-1}b\rangle$
14	$D_3{}^2$	36	[36,10]	$\langle a,b,c,d \mid a^3=b^2=c^3=d^2=1, bab=a^{-1},$ $ac=ca, ad=da, bc=cb, bd=db, dcd=c^{-1}\rangle$
15	$\mathbb{Z}_3\times A_4$	36	[36,11]	$\langle a,b,c,d \mid a^3=b^2=c^2=d^3=1, ab=ba,$ $ac=ca, ad=da, dbd^{-1}=bc=cb, dcd^{-1}=b\rangle$
16	$D_3\times \mathbb{Z}_6$	36	[36,12]	$\langle a,b,c \mid a^6=b^3=c^2=1,$ $ab=ba, ac=ca, cbc=b^{-1}\rangle$
17	$\mathbb{Z}_2\times \mathbb{Z}_3\rtimes S_3$	36	[36,13]	$\langle a,b,c,d \mid a^2=b^3=c^3=d^2=1, ab=ba,$ $ac=ca, ad=da, bc=cb, dbd=b^{-1}, dcd=c^{-1}\rangle$
18	$\mathbb{Z}_6{}^2$	36	[36,14]	$\langle a,b \mid a^6=b^6=1, ab=ba\rangle$
19	$\mathbb{Z}_{37}$	37	[37,1]	Abelian
20	D_{19}	38	[38,1]	$\langle a,b \mid a^{19}=b^2=1, bab=a^{-1}\rangle$

21	$\mathbb{Z}_{38}$	38	[38,2]	Abelian
22	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$	39	[39,1]	$\langle a, b \mid a^{13} = b^3 = 1, bab^{-1} = a^9 \rangle$
23	$\mathbb{Z}_{39}$	39	[39,2]	Abelian
24	$\mathbb{Z}_5 \rtimes_2 \mathbb{Z}_8$	40	[40,1]	$\langle a, b \mid a^5 = b^8 = 1, bab^{-1} = a^{-1} \rangle$
25	$\mathbb{Z}_{40}$	40	[40,2]	Abelian
26	$\mathbb{Z}_5 \rtimes \mathbb{Z}_8$	40	[40,3]	$\langle a, b \mid a^5 = b^8 = 1, bab^{-1} = a^3 \rangle$
27	Dic_{10}	40	[40,4]	$\langle a, b \mid a^{20} = 1, b^2 = a^{10}, bab^{-1} = a^{-1} \rangle$
28	$\mathbb{Z}_4 \times \mathrm{D}_5$	40	[40,5]	$\langle a, b, c \mid a^4 = b^5 = c^2 = 1,$ $ab = ba, ac = ca, cbc = b^{-1} \rangle$
29	D_{20}	40	[40,6]	$\langle a, b \mid a^{20} = b^2 = 1, bab = a^{-1} \rangle$
30	$\mathbb{Z}_2 \times \mathrm{Dic}_5$	40	[40,7]	$\langle a, b, c \mid a^2 = b^{10} = 1, c^2 = b^5,$ $ab = ba, ac = ca, cbc^{-1} = b^{-1} \rangle$
31	$\mathbb{Z}_5 \rtimes \mathrm{D}_4$	40	[40,8]	$\langle a, b, c \mid a^5 = b^4 = c^2 = 1,$ $bab^{-1} = cac = a^{-1}, cbc = b^{-1} \rangle$
32	$\mathbb{Z}_2 \times \mathbb{Z}_{20}$	40	[40,9]	$\langle a, b \mid a^2 = b^{20} = 1, ab = ba \rangle$
33	$\mathbb{Z}_5 \times \mathrm{D}_4$	40	[40,10]	$\langle a, b, c \mid a^5 = b^4 = c^2 = 1,$ $ab = ba, ac = ca, cbc = b^{-1} \rangle$
34	$\mathbb{Z}_5 \times \mathrm{Q}_8$	40	[40,11]	$\langle a, b, c \mid a^5 = b^4 = 1, c^2 = b^2,$ $ab = ba, ac = ca, cbc^{-1} = b^{-1} \rangle$
35	$\mathbb{Z}_2 \times \mathrm{F}_5$	40	[40,12]	$\langle a, b, c \mid a^2 = b^5 = c^4 = 1,$ $ab = ba, ac = ca, cbc^{-1} = b^3 \rangle$
36	$\mathbb{Z}_2^2 \times \mathrm{D}_5$	40	[40,13]	$\langle a, b, c, d \mid a^2 = b^2 = c^5 = d^2 = 1, ab = ba,$ $ac = ca, ad = da, bc = cb, bd = db, dcd = c^{-1} \rangle$
37	$\mathbb{Z}_2^2 \times \mathbb{Z}_{10}$	40	[40,14]	$\langle a, b, c \mid a^2 = b^2 = c^{10} = 1,$ $ab = ba, ac = ca, bc = cb \rangle$
38	$\mathbb{Z}_{41}$	41	[41,1]	Abelian
39	F_7	42	[42,1]	$\langle a, b \mid a^7 = b^6 = 1, bab^{-1} = a^5 \rangle$
40	$\mathbb{Z}_2 \times \mathbb{Z}_7 \rtimes \mathbb{Z}_3$	42	[42,2]	$\langle a, b, c \mid a^2 = b^7 = c^3 = 1,$ $ab = ba, ac = ca, cbc^{-1} = b^4 \rangle$
41	$\mathrm{D}_3 \times \mathbb{Z}_7$	42	[42,3]	$\langle a, b, c \mid a^7 = b^3 = c^2 = 1,$ $ab = ba, ac = ca, cbc = b^{-1} \rangle$
42	$\mathbb{Z}_3 \times \mathrm{D}_7$	42	[42,4]	$\langle a, b \mid a^{21} = b^2 = 1, bab = a^{-1} \rangle$
43	D_{21}	42	[42,5]	$\langle a, b \mid a^{21} = b^2 = 1, bab = a^{-1} \rangle$
44	$\mathbb{Z}_{42}$	42	[42,6]	Abelian
45	$\mathbb{Z}_{43}$	43	[43,1]	Abelian
46	Dic_{11}	44	[44,1]	$\langle a, b \mid a^{22} = 1, b^2 = a^{11}, bab^{-1} = a^{-1} \rangle$

47	$\mathbb{Z}_{44}$	44	[44,2]	Abelian
48	D_{22}	44	[44,3]	$\langle a,b \mid a^{22}=b^2=1, bab=a^{-1} \rangle$
49	$\mathbb{Z}_2\times\mathbb{Z}_{22}$	44	[44,4]	$\langle a,b \mid a^2=b^{22}=1, ab=ba \rangle$
50	$\mathbb{Z}_{45}$	45	[45,1]	Abelian
51	$\mathbb{Z}_3\times\mathbb{Z}_{15}$	45	[45,2]	$\langle a,b \mid a^3=b^{15}=1, ab=ba \rangle$
52	D_{23}	46	[46,1]	$\langle a,b \mid a^{23}=b^2=1, bab=a^{-1} \rangle$
53	$\mathbb{Z}_{46}$	46	[46,2]	Abelian
54	$\mathbb{Z}_{47}$	47	[47,1]	Abelian
55	$\mathbb{Z}_3\rtimes\mathbb{Z}_{16}$	48	[48,1]	$\langle a,b \mid a^3=b^{16}=1, bab^{-1}=a^{-1} \rangle$
56	$\mathbb{Z}_{48}$	48	[48,2]	Abelian
57	$\mathbb{Z}_4^2\rtimes\mathbb{Z}_3$	48	[48,3]	$\langle a,b,c \mid a^4=b^4=c^3=1, ab=ba,$ $cac^{-1}=ab^{-1}, cbc^{-1}=a^{-1}b^2 \rangle$
58	$D_3\times\mathbb{Z}_8$	48	[48,4]	$\langle a,b,c \mid a^8=b^3=c^2=1,$ $ab=ba, ac=ca, cbc=b^{-1} \rangle$
59	$\mathbb{Z}_8\rtimes S_3$	48	[48,5]	$\langle a,b,c \mid a^8=b^3=c^2=1, ab=ba,$ $cac=a^5, cbc=b^{-1} \rangle$
60	$\mathbb{Z}_{24}\rtimes\mathbb{Z}_2$	48	[48,6]	$\langle a,b \mid a^{24}=b^2=1, bab=a^{11} \rangle$
61	D_{24}	48	[48,7]	$\langle a,b \mid a^{24}=b^2=1, bab=a^{-1} \rangle$
62	Dic_{12}	48	[48,8]	$\langle a,b \mid a^{24}=1, b^2=a^{12}, bab^{-1}=a^{-1} \rangle$
63	$\mathbb{Z}_2\times\mathbb{Z}_3\rtimes\mathbb{Z}_8$	48	[48,9]	$\langle a,b,c \mid a^2=b^3=c^8=1,$ $ab=ba, ac=ca, cbc^{-1}=b^{-1} \rangle$
64	$\mathbb{Z}_4 . \mathrm{Dic}_3$	48	[48,10]	$\langle a,b,c \mid a^4=1, b^6=a^2, c^2=a^2b^3,$ $ab=ba, cac^{-1}=a^{-1}, cbc^{-1}=b^5 \rangle$
65	$\mathbb{Z}_4\times\mathrm{Dic}_3$	48	[48,11]	$\langle a,b,c \mid a^4=b^6=1, c^2=b^3,$ $ab=ba, ac=ca, cbc^{-1}=b^{-1} \rangle$
66	$\mathrm{Dic}_3\rtimes\mathbb{Z}_4$	48	[48,12]	$\langle a,b,c \mid a^6=c^4=1, b^2=a^3,$ $bab^{-1}=a^{-1}, ac=ca, cbc^{-1}=a^3b \rangle$
67	$\mathbb{Z}_4\rtimes\mathrm{Dic}_3$	48	[48,13]	$\langle a,b,c \mid a^4=b^6=1, c^2=b^3,$ $ab=ba, cac^{-1}=a^{-1}, cbc^{-1}=b^{-1} \rangle$
68	$D_6\rtimes\mathbb{Z}_4$	48	[48,14]	$\langle a,b,c \mid a^6=b^2=c^4=1, bab=a^{-1},$ $ac=ca, cbc^{-1}=a^3b \rangle$
69	$D_4\rtimes S_3$	48	[48,15]	$\langle a,b,c,d \mid a^4=b^2=c^3=d^2=1,$ $bab=dad=a^{-1}, ac=ca, bc=cb,$ $dbd=ab, dcd=c^{-1} \rangle$

70	$D_4.S_3$	48	[48,16]	$\langle a,b,c,d \mid a^4=b^2=c^3=1, d^2=a^2,$ $bab=dad^{-1}=a^{-1}, ac=ca, bc=cb,$ $dbd^{-1}=ab, dcd^{-1}=c^{-1}\rangle$
71	$Q_8\rtimes_2 S_3$	48	[48,17]	$\langle a,b,c,d \mid a^4=c^3=d^2=1, b^2=a^2,$ $bab^{-1}=dad=a^{-1}, ac=ca, bc=cb,$ $dbd=a^{-1b}, dcd=c^{-1}\rangle$
72	$\mathbb{Z}_3\rtimes Q_{16}$	48	[48,18]	$\langle a,b,c \mid a^3=b^8=1, c^2=b^4,$ $bab^{-1}=a^{-1}, ac=ca, cbc^{-1}=b^{-1}\rangle$
73	$\mathbb{Z}_6.D_4$	48	[48,19]	$\langle a,b,c \mid a^6=b^4=1,\ c^2=a^3, bab^{-1}=cac^{-1}=a^{-1}, cbc^{-1}=a^3b^{-1}\rangle$
74	$\mathbb{Z}_4\times\mathbb{Z}_{12}$	48	[48,20]	$\langle a,b \mid a^4=b^{12}=1, ab=ba\rangle$
75	$\mathbb{Z}_3\times\mathbb{Z}_2{}^2\rtimes\mathbb{Z}_4$	48	[48,21]	$\langle a,b,c,d \mid a^3=b^2=c^2=d^4=1, ab=ba,$ $ac=ca, ad=da, dbd^{-1}=bc=cb, cd=dc\rangle$
76	$\mathbb{Z}_3\times\mathbb{Z}_4\rtimes\mathbb{Z}_4$	48	[48,22]	$\langle a,b,c \mid a^3=b^4=c^4=1,$ $ab=ba, ac=ca, cbc^{-1}=b^{-1}\rangle$
77	$\mathbb{Z}_2\times\mathbb{Z}_{24}$	48	[48,23]	$\langle a,b \mid a^2=b^{24}=1, ab=ba\rangle$
78	$\mathbb{Z}_3\times M_4(2)$	48	[48,24]	$\langle a,b,c \mid a^3=b^8=c^2=1,$ $ab=ba, ac=ca, cbc=b^5\rangle$
79	$\mathbb{Z}_3\times D_8$	48	[48,25]	$\langle a,b,c \mid a^3=b^8=c^2=1,$ $ab=ba, ac=ca, cbc=b^{-1}\rangle$
80	$\mathbb{Z}_3\times SD_{16}$	48	[48,26]	$\langle a,b,c \mid a^3=b^8=c^2=1,$ $ab=ba, ac=ca, cbc=b^3\rangle$
81	$\mathbb{Z}_3\times Q_{16}$	48	[48,27]	$\langle a,b,c \mid a^3=b^8=1, c^2=b^4,$ $ab=ba, ac=ca, cbc^{-1}=b^{-1}\rangle$
82	$CSU_2(\mathbb{F}_3)$	48	[48,28]	$\langle a,b,c,d \mid a^4=c^3=1,\ b^2=d^2=a^2, bab^{-1}=dbd^{-1}=a^{-1}, cac^{-1}$ $=ab, dad^{-1}=a^2b, cbc^{-1}=a, dcd^{-1}=c^{-1}\rangle$
83	$GL_2(\mathbb{F}_3)$	48	[48,29]	$\langle a,b,c,d \mid a^4=c^3=d^2=1,\ b^2=a^2, bab^{-1}=dbd=a^{-1}, cac^{-1}$ $=ab, dad=a^2b, cbc^{-1}=a, dcd=c^{-1}\rangle$
84	$\underline{A_4\rtimes\mathbb{Z}_4}$	48	[48,30]	$\langle a,b,c,d \mid a^2=b^2=c^3=d^4=1, cac^{-1}=dad^{-1}=ab=ba, cbc^{-1}$ $=a, bd=db, dcd^{-1}=c^{-1}\rangle$
85	$\mathbb{Z}_4\times A_4$	48	[48,31]	$\langle a,b,c \mid a^2=b^{12}=1, c^2=b^6,$ $ab=ba, ac=ca, cbc^{-1}=b^{-1}\rangle$
86	$\mathbb{Z}_2\times SL_2(\mathbb{F}_3)$	48	[48,32]	$\langle a,b,c,d \mid a^2=b^4=d^3=1, c^2=b^2, ab=ba, ac=ca, ad=da, cbc^{-1}$ $=b^{-1}, dbd^{-1}=c, dcd^{-1}=bc\rangle$
87	$\mathbb{Z}_4.A_4$	48	[48,33]	$\langle a,b,c,d \mid a^4=d^3=1, b^2=c^2=a^2, ab=ba, ac=ca, ad=da, cbc^{-1}$ $=a^2b, dbd^{-1}=a^2bc, dcd^{-1}=b\rangle$
88	$\mathbb{Z}_2\times Dic_6$	48	[48,34]	$\langle a,b,c,d \mid a^2=b^4=c^3=d^2=1, ab=ba,$ $ac=ca, ad=da, bc=cb, bd=db, dcd=c^{-1}\rangle$
89	$D_3\times\mathbb{Z}_2\times\mathbb{Z}_4$	48	[48,35]	$\langle a,b,c \mid a^2=b^{12}=c^2=1, ab=ba,$ $ac=ca, cbc=b^{-1}\rangle$
90	$\mathbb{Z}_2\times D_{12}$	48	[48,36]	$\langle a,b,c \mid a^4=c^2=1, b^6=a^2,$ $ab=ba, ac=ca, cbc=a^2b^5\rangle$
91	$\mathbb{Z}_4\circ D_{12}$	48	[48,37]	$\langle a,b,c,d \mid a^4=b^2=c^3=d^2=1,$ $bab=a^{-1}, ac=ca, ad=da, bc=cb,$ $dbd=a^2b, dcd=c^{-1}\rangle$
92	$S_3\times D_4$	48	[48,38]	$\langle a,b,c,d \mid a^3=b^2=c^4=d^2=1, bab=a^{-1}, ac=ca, ad=da, bc$ $=cb, bd=db, dcd=c^{-1}\rangle$
93	$D_4\rtimes_2 S_3$	48	[48,39]	$\langle a,b,c,d \mid a^3=b^2=c^4=1, d^2=c^2,$

				$bab = a^{-1}, ac = ca, ad = da, bc = cb,$ $bd = db, dcd^{-1} = c^{-1} \rangle$
94	$S_3 \times Q_8$	48	[48,40]	$\langle a, b, c, d \mid a^2 = b^2 = c^6 = 1,$ $d^2 = c^3, ab = ba, ac = ca, ad = da,$ $bc = cb, bd = db, dcd^{-1} = c^{-1} \rangle$
95	$Q_8 \rtimes_3 S_3$	48	[48,41]	$\langle a, b, c, d \mid a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dad = a^{-1}, ac = ca, bc = cb, bd = db, dcd = c^{-1} \rangle$
96	$\mathbb{Z}_2^2 \times Dic_3$	48	[48,42]	$\langle a, b, c, d \mid a^2 = b^2 = c^6 = 1, d^2 = c^3, ab = ba, ac = ca, ad = da, bc = cb, bd = db, dcd^{-1} = c^{-1} \rangle$
97	$\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes D_4$	48	[48,43]	$\langle a, b, c, d \mid a^2 = b^3 = c^4 = d^2 = 1,$ $ab = ba, ac = ca, ad = da,$ $cbc^{-1} = dbd = b^{-1}, dcd = c^{-1} \rangle$
98	$\mathbb{Z}_2^2 \times \mathbb{Z}_{12}$	48	[48,44]	$\langle a, b, c \mid a^2 = b^2 = c^{12} = 1,$ $ab = ba, ac = ca, bc = cb \rangle$
99	$\mathbb{Z}_6 \times D_4$	48	[48,45]	$\langle a, b, c \mid a^6 = b^4 = c^2 = 1,$ $ab = ba, ac = ca, cbc = b^{-1} \rangle$
100	$\mathbb{Z}_6 \times Q_8$	48	[48,46]	$\langle a, b, c \mid a^6 = b^4 = 1, c^2 = b^2, ab = ba,$ $ac = ca, cbc^{-1} = b \rangle$
101	$\mathbb{Z}_3 \times \mathbb{Z}_4 \circ D_4$	48	[48,47]	$\langle a, b, c, d \mid a^3 = b^4 = d^2 = 1, c^2 = b^2,$ $ab = ba, ac = ca, ad = da, bc = cb,$ $bd = db, dcd = b^2c \rangle$
102	$\mathbb{Z}_2 \times D_4$	48	[48,48]	$\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^3 = 1,$ $ab = ba, ac = ca, ad = da, ae = ea, bc = cb,$ $bd = db, be = eb, ece^{-1} = cd = dc,$ $ede^{-1} = c \rangle$
103	$\mathbb{Z}_2^2 \times A_4$	48	[48,49]	$\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^3 = e^2 = 1, ab = ba, ac = ca, ad = da, ae = ea, dbd^{-1} = ebe = bc = cb, dcd^{-1} = b, ce = ec, ede = d^{-1} \rangle$
104	$\mathbb{Z}_2^2 \rtimes A_4$	48	[48,50]	$\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^3 = 1,$ $eae^{-1} = ab = ba, ac = ca, ad = da, bc = cb,$ $bd = db, ebe^{-1} = a, ece^{-1} = cd = dc,$ $ede^{-1} = c \rangle$
105	$D_3 \times \mathbb{Z}_2^3$	48	[48,51]	$\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^3 = e^2 = 1$ $, ab = ba, ac = ca, ad = da, ae = ea, bc = cb,$ $bd = db, be = eb, cd = dc, ce = ec,$ $ede = d^{-1} \rangle$
106	$\mathbb{Z}_2^3 \times \mathbb{Z}_6$	48	[48,52]	$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^6 = 1, ab = ba,$ $ac = ca, ad = da, bc = cb, bd = db, cd = dc \rangle$
107	$\mathbb{Z}_{49}$	49	[49,1]	Abelian
108	$\mathbb{Z}_7^2$	49	[49,2]	$\langle a, b \mid a^7 = b^7 = 1, ab = ba \rangle$
109	D_{25}	50	[50,1]	$\langle a, b \mid a^{25} = b^2 = 1, bab = a^{-1} \rangle$
110	$\mathbb{Z}_{50}$	50	[50,2]	Abelian
111	$\mathbb{Z}_5 \times D_5$	50	[50,3]	$\langle a, b, c \mid a^5 = b^5 = c^2 = 1,$ $ab = ba, ac = ca, cbc = b^{-1} \rangle$
112	$\mathbb{Z}_5 \rtimes D_5$	50	[50,4]	$\langle a, b, c \mid a^5 = b^5 = c^2 = 1,$ $ab = ba, cac = a^{-1}, cbc = b^{-1} \rangle$
113	$\mathbb{Z}_5 \times \mathbb{Z}_{10}$	50	[50,5]	$\langle a, b \mid a^5 = b^{10} = 1, ab = ba \rangle$
114	$\mathbb{Z}_{51}$	51	[51,1]	Abelian

115	Dic_{13}	52	[52,1]	$\langle a,b \mid a^{26}=1, b^2=a^{13}, bab^{-1}=a^{-1} \rangle$
116	$\mathbb{Z}_{52}$	52	[52,2]	Abelian
117	$\mathbb{Z}_{13}\rtimes \mathbb{Z}_4$	52	[52,3]	$\langle a,b \mid a^{13}=b^4=1, bab^{-1}=a^5 \rangle$
118	D_{26}	52	[52,4]	$\langle a,b \mid a^{26}=b^2=1, bab=a^{-1} \rangle$
119	$\mathbb{Z}_2\times \mathbb{Z}_{26}$	52	[52,5]	$\langle a,b \mid a^2=b^{26}=1, ab=ba \rangle$
120	$\mathbb{Z}_{53}$	53	[53,1]	Abelian
121	D_{27}	54	[54,1]	$\langle a,b \mid a^{27}=b^2=1, bab=a^{-1} \rangle$
122	$\mathbb{Z}_{54}$	54	[54,2]	Abelian
123	$\mathbb{Z}_3\times \mathrm{D}_9$	54	[54,3]	$\langle a,b,c \mid a^3=b^9=c^2=1,$ $ab=ba, ac=ca, cbc=b^{-1} \rangle$
124	$\mathrm{D}_3\times \mathbb{Z}_9$	54	[54,4]	$\langle a,b,c \mid a^9=b^3=c^2=1,$ $ab=ba, ac=ca, cbc=b^{-1} \rangle$
125	$\mathbb{Z}_3^2\rtimes \mathbb{Z}_6$	54	[54,5]	$\langle a,b,c \mid a^3=b^3=c^6=1,$ $ab=ba, cac^{-1}=a^{-1}b^{-1}, cbc^{-1}=b^{-1} \rangle$
126	$\mathbb{Z}_9\rtimes \mathbb{Z}_6$	54	[54,6]	$\langle a,b \mid a^9=b^6=1, bab^{-1}=a^2 \rangle$
127	$\mathbb{Z}_9\rtimes \mathrm{S}_3$	54	[54,7]	$\langle a,b,c \mid a^9=b^3=c^2=1,$ $ab=ba, cac=a^{-1}, cbc=b^{-1} \rangle$
128	$\mathrm{He}_3\rtimes_2 \mathbb{Z}_2$	54	[54,8]	$\langle a,b,c,d \mid a^3=b^3=c^3=d^2=1, ab=ba, cac^{-1}=ab^{-1}, dad=a^{-1}, bc$ $=cb, bd=db, dcd=c^{-1}$
129	$\mathbb{Z}_3\times \mathbb{Z}_{18}$	54	[54,9]	$\langle a,b \mid a^3=b^{18}=1, ab=ba \rangle$
130	$\mathbb{Z}_2\times \mathrm{He}_3$	54	[54,10]	$\langle a,b,c,d \mid a^2=b^3=c^3=d^3=1, ab=ba,$ $ac=ca, ad=da, bc=cb, dbd^{-1}=bc^{-1},$ $cd=dc \rangle$
131	$\mathbb{Z}_2\times 3_{-}^{1+2}$	54	[54,11]	$\langle a,b,c \mid a^2=b^9=c^3=1,$ $ab=ba, ac=ca, cbc^{-1}=b^4 \rangle$
132	$\mathrm{D}_3\times \mathbb{Z}_3^2$	54	[54,12]	$\langle a,b,c,d \mid a^3=b^3=c^3=d^2=1, ab=ba,$ $ac=ca, ad=da, bc=cb, bd=db, dcd=c^{-1} \rangle$
133	$\mathbb{Z}_3\times \mathbb{Z}_3\rtimes \mathrm{S}_3$	54	[54,13]	$\langle a,b,c,d \mid a^3=b^3=c^3=d^2=1,$ $ab=ba, ac=ca, ad=da, bc=cb,$ $dbd=b^{-1}, dcd=c^{-1} \rangle$
134	$\mathbb{Z}_3^3\rtimes \mathbb{Z}_2$	54	[54,14]	$\langle a,b,c,d \mid a^3=b^3=c^3=d^2=1,$ $ab=ba, ac=ca, dad=a^{-1}, bc=cb,$ $dbd=b^{-1}, dcd=c^{-1} \rangle$
135	$\mathbb{Z}_3^2\times \mathbb{Z}_6$	54	[54,15]	$\langle a,b,c \mid a^3=b^3=c^6=1,$ $ab=ba, ac=ca, bc=cb \rangle$
136	$\mathbb{Z}_{11}\rtimes \mathbb{Z}_5$	55	[55,1]	$\langle a,b \mid a^{11}=b^5=1, bab^{-1}=a^3 \rangle$
137	$\mathbb{Z}_{55}$	55	[55,2]	Abelian
138	$\mathbb{Z}_7\rtimes \mathbb{Z}_8$	56	[56,1]	$\langle a,b \mid a^7=b^8=1, bab^{-1}=a^{-1} \rangle$
139	$\mathbb{Z}_{56}$	56	[56,2]	Abelian
140	Dic_{14}	56	[56,3]	$\langle a,b \mid a^{28}=1, b^2=a^{14}, bab^{-1}=a^{-1} \rangle$
141	$\mathbb{Z}_4\times \mathrm{D}_7$	56	[56,4]	$\langle a,b,c \mid a^4=b^7=c^2=1,$ $ab=ba, ac=ca, cbc=b^{-1} \rangle$
142	D_{28}	56	[56,5]	$\langle a,b \mid a^{28}=b^2=1, bab=a^{-1} \rangle$
143	$\mathbb{Z}_2\times \mathrm{Dic}_7$	56	[56,6]	$\langle a,b,c \mid a^2=b^{14}=1, c^2=b^7,$ $ab=ba, ac=ca, cbc^{-1}=b^{-1} \rangle$
144	$\mathbb{Z}_7\rtimes \mathrm{D}_4$	56	[56,7]	$\langle a,b,c \mid a^7=b^4=c^2=1,$ $bab^{-1}=cac=a^{-1}, cbc=b^{-1} \rangle$
145	$\mathbb{Z}_2\times \mathbb{Z}_{28}$	56	[56,8]	$\langle a,b \mid a^2=b^{28}=1, ab=ba \rangle$
146	$\mathbb{Z}_7\times \mathrm{D}_4$	56	[56,9]	$\langle a,b,c \mid a^7=b^4=c^2=1,$

				$ab = ba, ac = ca, cbc = b^{-1} \rangle$
147	$\mathbb{Z}_7 \times Q_8$	56	[56,10]	$\langle a, b, c \mid a^7 = b^4 = 1, c^2 = b^2,$ $ab = ba, ac = ca, cbc^{-1} = b^{-1} \rangle$
148	F_8	56	[56,11]	$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^7 = 1,$ $ab = ba, ac = ca, dad^{-1} = cb = bc,$ $dbd^{-1} = a, dcd^{-1} = b \rangle$
149	$\mathbb{Z}_2^2 \times D_7$	56	[56,12]	$\langle a, b, c, d \mid a^2 = b^2 = c^7 = d^2 = 1, ab = ba,$ $ac = ca, ad = da, bc = cb, bd = db, dcd = c^{-1} \rangle$
150	$\mathbb{Z}_2^2 \times \mathbb{Z}_{14}$	56	[56,13]	$\langle a, b, c \mid a^2 = b^2 = c^{14} = 1, ab = ba,$ $ac = ca, bc = cb \rangle$
151	$\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$	57	[57,1]	$\langle a, b \mid a^{19} = b^3 = 1, bab^{-1} = a^{11} \rangle$
152	$\mathbb{Z}_{57}$	57	[57,2]	Abelian
153	D_{29}	58	[58,1]	$\langle a, b \mid a^{29} = b^2 = 1, bab = a^{-1} \rangle$
154	$\mathbb{Z}_{58}$	58	[58,2]	Abelian
155	$\mathbb{Z}_{59}$	59	[59,1]	Abelian
156	$\mathbb{Z}_5 \times Dic_3$	60	[60,1]	$\langle a, b, c \mid a^5 = b^6 = 1, c^2 = b^3,$ $ab = ba, ac = ca, cbc^{-1} = b^{-1} \rangle$
157	$\mathbb{Z}_3 \times Dic_5$	60	[60,2]	$\langle a, b, c \mid a^3 = b^{10} = 1, c^2 = b^5,$ $ab = ba, ac = ca, cbc^{-1} = b^{-1} \rangle$
158	Dic_{15}	60	[60,3]	$\langle a, b \mid a^{30} = 1, b^2 = a^{15}, bab^{-1} = a^{-1} \rangle$
159	$\mathbb{Z}_{60}$	60	[60,4]	Abelian
160	A_5	60	[60,5]	$\langle a, b \mid a^5 = b^3 = ab^2 = 1 \rangle$
161	$\mathbb{Z}_3 \times F_5$	60	[60,6]	$\langle a, b, c \mid a^3 = b^5 = c^4 = 1,$ $ab = ba, ac = ca, cbc^{-1} = b^3 \rangle$
162	$\mathbb{Z}_3 \rtimes F_5$	60	[60,7]	$\langle a, b, c \mid a^3 = b^5 = c^4 = 1,$ $ab = ba, cac^{-1} = a^{-1}, cbc^{-1} = b^3 \rangle$
163	$D_3 \times D_5$	60	[60,8]	$\langle a, b, c, d \mid a^3 = b^2 = c^5 = d^2 = 1,$ $bab = a^{-1}, ac = ca, ad = da,$ $bc = cb, bd = db, dcd = c^{-1} \rangle$
164	$\mathbb{Z}_5 \times A_4$	60	[60,9]	$\langle a, b, c, d \mid a^5 = b^2 = c^2 = d^3 = 1,$ $ab = ba, ac = ca, ad = da,$ $dbd^{-1} = bc = cb, dcd^{-1} = b \rangle$
165	$\mathbb{Z}_6 \times D_5$	60	[60,10]	$\langle a, b, c \mid a^6 = b^5 = c^2 = 1,$ $ab = ba, ac = ca, cbc = b^{-1} \rangle$
166	$D_3 \times Z_{10}$	60	[60,11]	$\langle a, b, c \mid a^{10} = b^3 = c^2 = 1, ab = ba, ac = ca, cbc = b^{-1} \rangle$
167	D_{30}	60	[60,12]	$\langle a, b \mid a^{30} = b^2 = 1, bab = a^{-1} \rangle$
168	$\mathbb{Z}_2 \times \mathbb{Z}_{30}$	60	[60,13]	$\langle a, b \mid a^2 = b^{30} = 1, ab = ba \rangle$

***Table* 2.1** *All groups of order* 33 *to* 60 *with their group presentation* [9]

The group of order from 33 to 60 listed in **Table 2.1** is separated into a few partitions as follows.

Partition	No. of Group from Table 2.1
A	1,2,4,6,19,21,23,25,38,44,45,47,50,53,54,56,107,110,114,116,120,122,137, 139,152,154,155,159
B	9,12,18,32,37,49,51,74,77,98,106,108,113,119,129,135,145,150,168
C	3,8,20,29,43,48,52,61,109,118,121,142,153,167
D	14,163
E	16,28,33,36,41,42,58,78,89,90,99,102,105,106,111,123,124,132,141,146, 149,165,166

F	5,7,10,11,13,15,17,22,24,26,27,30,31,34,35,39,40,42,46,55,57,59,60,62,63, 64,65,66,67,68,69,70,71,72,73,75,76,78,80,81,82,83,84,85,86,87,88,91,92, 93,94,95,96,97,100,101,103,104,112,115,117,125,126,127,128,130,131,133, 134,136,138,140,143,144,147,148,151,156,157,158,160,161.162,164

***Table* 2.2** *No. of Group from Table* 2.1 *separated by Partition*

2. THE DETERMINATION OF METABELIAN GROUPS OF ORDER 33 TO 60

In 2012, Abdul Rahman and Sarmin [1] conducted a study to identify all metabelian groups of order at most 24. Their research catalogued 59 groups of order less than 24 and 15 groups of order 24, including both abelian and nonabelian groups. The study revealed that all groups of order less than 24 are metabelian. However, among the groups of order 24, two were identified as non-metabelian, namely S_4 and $\mathbb{S}l(2,3)$. Later in 2017, Simon [2] extended this research to include metabelian groups up to the order of 32. This study listed 70 groups ranging from orders 25 to 32, again including both abelian and nonabelian groups. The findings showed that all groups within this range are metabelian. Therefore, the focus of this proceeding is to determine the metabelian groups for orders 33 to 60.

Theorem 3.1

Let *G* be a group in Partition A. Then *G* is metabelian. □

Proof

In Partition A, all group listed are cyclic. Hence, by Theorem 2.8, every cyclic group is abelian and Theorem 2.3, every abelian group is metabelian, those groups are metabelian. ■

Theorem 3.2

Let *G* be a group in Partition B. Then *G* is metabelian. □

Proof

In Partition B, each of the group listed is a direct product of two abelian groups in the form ${\mathbb{Z}_m}^k \times \mathbb{Z}_n$ for some $m, n, k \in \mathbb{Z}^+$. Hence, by Theorem 2.3, every abelian group is metabelian and Theorem 2.10, those groups are metabelian. ■

Theorem 3.3

Let *G* be a group in Partition C. Then *G* is metabelian. □

Proof

In Partition C, all group listed are dihedral group, hence, by Theorem 2.11, any dihedral group is metabelian, those groups are metabelian.

Theorem 3.4

Let *G* be a group in Partition D. Then *G* is metabelian. □

Proof

In Partition D, the group listed are a direct product of two dihedral groups in the form

$D_m \times D_n$ for some $m, n \geq 3$. Hence, by Theorem 2.11, any dihedral group is metabelian and Theorem 2.10, the direct product of metabelian group is metabelian, those groups are metabelian. ■

Theorem 3.5

Let *G* be a group in Partition E. Then *G* is metabelian. □

Proof

In Partition E, the group listed are a direct product of abelian group and dihedral group in the form $\mathbb{Z}_m{}^k \times D_n$ for some $m, k \in \mathbb{Z}^+, n \geq 3$. Hence, by Theorem 2.3 (every abelian group is metabelian), Theorem 2.11(any dihedral group is metabelian) and Theorem 2.10, those groups are metabelian. ■

3. THE DETERMINATION OF DERIVED SUBGROUP OF SOME GROUPS OF ORDER 33 TO 60 USING GAP SOFTWARE.

In this subsection, let *G* be a group in Partition F is order from 36 to 60. The data are taken from [9]. Then all metabelian groups among those are determined. The following algorithms used in determining the derived subgroup of some groups of order at most 60. First of all, the groups Id have to be defined in Groups, Algorithms and Programming (GAP) software. Thus, in this research GAP is used in determination of derived subgroup and the derived length of the group of order from 36 to 60 as code programming below. For an example, the group Id for generalized Frobenius group of order 42, F_7 is [42,1] as given in [9].

```
gap> G:=SmallGroup(42,1);
<pc group of size 42 with 3 generators>
gap> d:=DerivedSubgroup(G);
Group([ f3 ])
gap> StructureDescription(d);
"C7"
gap> DerivedLength(G);
2
```

Line 1: `gap> G:=SmallGroup(42,1);`
`gap>`: This is the GAP prompt, indicating that the user is entering a command into the GAP system.
`G:=`: This assigns the result of the following command to the variable G.
`G:=SmallGroup(42,1)`: Creates a specific small group of order 42. The parameters `42` and `1` specify that the first group of order 42 in GAP's small group library is desired.

Line 2: `<pc group of size 42 with 3 generators>`
`<pc group of size 42 with 3 generators>`: GAP confirms that it has created a polycyclic (pc) group of order 42 with 3 generators.

Line 3: `gap> d:=DerivedSubgroup(G);`
`d:=`:This assigns the result of the following command to the variable d.
`DerivedSubgroup(G)`: This function computes the derived subgroup (also known as the commutator subgroup) of the group *G*. The derived subgroup is the subgroup generated by all the commutators of *G*. Commutators measure how far the group is from being abelian.

Line 4: `Group([ f3 ])`
`Group([ f3 ])`: GAP confirms that the derived subgroup d is generated by the element `f3`. This means that `f3` is a generator of the derived subgroup.

Line 5: `gap> StructureDescription(d);`
`StructureDescription(d)`: This function returns a human-readable description of the structure of the subgroup d.

Line 6: `"C7"`
`"C7"`: GAP indicates that the structure of the derived subgroup d is the cyclic group of order 7. This means that the derived subgroup is isomorphic to the cyclic group of order 7.

Line 7: `gap> DerivedLength(G);`
`DerivedLength(G)`: Calculates the derived length of the group *G*. The derived length is a measure of how many steps are needed to reach the trivial subgroup by repeatedly taking derived subgroups. It indicates how many times the commutator subgroup needs to be taken to get an abelian group (where further commutators are trivial).

Line 8: `2`
2: Indicates that the derived length of *G* is 2. By definition 2.9, when derived length $\leq 2 \leftrightarrow G$ is metabelian group. This means that *G* is a solvable group and that its commutator subgroup's commutator subgroup is the trivial group. Specifically, *G* is a metabelian group (a group whose derived subgroup is abelian).

Observation 3.1
Let *G* be a group of order from 33 to 60 with groups are Dic_9, $\mathbb{Z}_3 . A_4$, $\mathbb{Z}_3 \times Dic_3$, $\mathbb{Z}_3 \rtimes Dic_3$, $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes S_3$, $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$, $\mathbb{Z}_5 \rtimes_2 \mathbb{Z}_8$, $\mathbb{Z}_5 \rtimes \mathbb{Z}_8$, Dic_{10}, $\mathbb{Z}_2 \times Dic_5$, $\mathbb{Z}_5 \rtimes D_4$, $\mathbb{Z}_5 \times Q_8$, $\mathbb{Z}_2 \times F_5$, F_7, $\mathbb{Z}_2 \times \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, Dic_{11}, $\mathbb{Z}_3 \rtimes \mathbb{Z}_{16}$, $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_8 \rtimes S_3$, $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$, Dic_{12}, $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes \mathbb{Z}_8$, $\mathbb{Z}_4 . Dic_3$, $\mathbb{Z}_4 \times Dic_3$, $Dic_3 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_4 \rtimes Dic_3$, $D_6 \rtimes \mathbb{Z}_4$, $D_4 \rtimes S_3$, $D_4 . S_3$, $Q_8 \rtimes_2 S_3$, $\mathbb{Z}_3 \rtimes Q_{16}$, $\mathbb{Z}_6 . D_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3 \times M_4(2)$, $\mathbb{Z}_3 \times SD_{16}$, $\mathbb{Z}_3 \times Q_{16}$, $\mathbb{Z}_4 \times A_4$, $\mathbb{Z}_2 \times Dic_6$, $\mathbb{Z}_4 \circ D_{12}$, $S_3 \times D_4$, $D_4 \rtimes_2 S_3$, $S_3 \times Q_8$, $Q_8 \rtimes_3 S_3$, $\mathbb{Z}_2^2 \times Dic_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes D_4$, $\mathbb{Z}_6 \times Q_8$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \circ D_4$, $\mathbb{Z}_2^2 \times A_4$, $\mathbb{Z}_2^2 \rtimes A_4$, $\mathbb{Z}_5 \rtimes D_5$, Dic_{13}, $\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_6$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_6$, $\mathbb{Z}_9 \rtimes S_3$, $\mathbb{Z}_2 \times He_3$, $\mathbb{Z}_2 \times 3_-^{1+2}$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \rtimes S_3$, $\mathbb{Z}_3^3 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_8$, Dic_{14}, $\mathbb{Z}_2 \times Dic_7$, $\mathbb{Z}_7 \rtimes D_4$, $\mathbb{Z}_7 \times Q_8$, F_8, $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$, $\mathbb{Z}_5 \times Dic_3$, $\mathbb{Z}_3 \times Dic_5$, Dic_{15}, A_5, $\mathbb{Z}_3 \times F_5$, $\mathbb{Z}_3 \rtimes F_5$ or $\mathbb{Z}_5 \times A_4$. Then, *G* has derived length 2. In contrast, groups are $\underline{A_4 \rtimes \mathbb{Z}_4}$, $\mathbb{Z}_2 \times SL_2(\mathbb{F}_3)$, $\underline{\mathbb{Z}_4 . A_4}$, or $He_3 \rtimes_2 \mathbb{Z}_2$ have a derived length of 3 and for groups are $CSU_2(\mathbb{F}_3)$ or $GL_2(\mathbb{F}_3)$ have a derived length of 4.

Proof
Suppose *G* is equal to groups are Dic_9, $\mathbb{Z}_3 . A_4$, $\mathbb{Z}_3 \times Dic_3$, $\mathbb{Z}_3 \rtimes Dic_3$, $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes S_3$, $\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$, $\mathbb{Z}_5 \rtimes_2 \mathbb{Z}_8$, $\mathbb{Z}_5 \rtimes \mathbb{Z}_8$, Dic_{10}, $\mathbb{Z}_2 \times Dic_5$, $\mathbb{Z}_5 \rtimes D_4$, $\mathbb{Z}_5 \times Q_8$, $\mathbb{Z}_2 \times F_5$, F_7, $\mathbb{Z}_2 \times \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, Dic_{11}, $\mathbb{Z}_3 \rtimes \mathbb{Z}_{16}$, $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_8 \rtimes S_3$, $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$, Dic_{12}, $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes \mathbb{Z}_8$, $\mathbb{Z}_4 . Dic_3$, $\mathbb{Z}_4 \times Dic_3$, $Dic_3 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_4 \rtimes Dic_3$, $D_6 \rtimes \mathbb{Z}_4$, $D_4 \rtimes S_3$, $D_4 . S_3$, $Q_8 \rtimes_2 S_3$, $\mathbb{Z}_3 \rtimes Q_{16}$, $\mathbb{Z}_6 . D_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3 \times M_4(2)$, $\mathbb{Z}_3 \times SD_{16}$, $\mathbb{Z}_3 \times Q_{16}$, $\mathbb{Z}_4 \times A_4$, $\mathbb{Z}_2 \times Dic_6$, $\mathbb{Z}_4 \circ D_{12}$, $S_3 \times D_4$, $D_4 \rtimes_2 S_3$, $S_3 \times Q_8$, $Q_8 \rtimes_3 S_3$, $\mathbb{Z}_2^2 \times Dic_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes D_4$, $\mathbb{Z}_6 \times Q_8$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \circ D_4$, $\mathbb{Z}_2^2 \times A_4$, $\mathbb{Z}_2^2 \rtimes A_4$, $\mathbb{Z}_5 \rtimes D_5$, Dic_{13}, $\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$, $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_6$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_6$, $\mathbb{Z}_9 \rtimes S_3$, $\mathbb{Z}_2 \times He_3$, $\mathbb{Z}_2 \times 3_-^{1+2}$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \rtimes S_3$, $\mathbb{Z}_3^3 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_8$, Dic_{14}, $\mathbb{Z}_2 \times Dic_7$, $\mathbb{Z}_7 \rtimes D_4$, $\mathbb{Z}_7 \times Q_8$, F_8, $\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$, $\mathbb{Z}_5 \times Dic_3$, $\mathbb{Z}_3 \times Dic_5$, Dic_{15}, A_5, $\mathbb{Z}_3 \times F_5$, $\mathbb{Z}_3 \rtimes F_5$ or $\mathbb{Z}_5 \times A_4$. By using GAP, then the derived length is 2. By definition 2.9, thus *G* is metabelian. ■

Conjecture 3.1
Let *G* be equal to groups are $CSU_2(\mathbb{F}_3)$, $GL_2(\mathbb{F}_3)$, $\underline{A_4 \rtimes \mathbb{Z}_4}$, $\mathbb{Z}_2 \times SL_2(\mathbb{F}_3)$, $\underline{\mathbb{Z}_4 . A_4}$ or $He_3 \rtimes_2 \mathbb{Z}_2$. Then, *G* is not metabelian. □

Conclusion

With the scope of this research, 162 from 168 groups of order from 33 to 60 are detected as metabelian groups and the rest five groups of order 48 and one group of order 54 are not metabelian which are

1. $CSU_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = 1,\ b^2 = d^2 = a^2, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = ab, dad^{-1} = a^2 b, cbc^{-1} = a, dcd^{-1} = c^{-1} \rangle$
2. $GL_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = d^2 = 1,\ b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab, dad = a^2 b, cbc^{-1} = a, dcd = c^{-1} \rangle$
3. $\underline{A_4 \rtimes \mathbb{Z}_4} = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^4 = 1, cac^{-1} = dad^{-1} = ab = ba, cbc^{-1} = a, bd = db, dcd^{-1} = c^{-1} \rangle$
4. $\mathbb{Z}_2 \times SL_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^2 = b^4 = d^3 = 1, c^2 = b^2, ab = ba, ac = ca, ad = da, cbc^{-1} = b^{-1}, dbd^{-1} = c, dcd^{-1} = bc \rangle$
5. $\underline{\mathbb{Z}_4 . A_4} = \langle a, b, c, d \mid a^4 = d^3 = 1, b^2 = c^2 = a^2, ab = ba, ac = ca, ad = da, cbc^{-1} = a^2 b, dbd^{-1} = a^2 bc, dcd^{-1} = b \rangle$
6. $He_3 \rtimes_2 \mathbb{Z}_2 = \langle a, b, c, d \mid a^3 = b^3 = c^3 = d^2 = 1, ab = ba, cac^{-1} = ab^{-1}, dad = a^{-1}, bc = cb, bd = db, dcd = c^{-1} \rangle$

All groups of order 33 to 60 have been proved as metabelian groups or not based on their group presentations. The Groups, Algorithms and Programming (GAP) software has been used to facilitate some of the computations and proofs.

Acknowledgement

The researcher would like to thank all people who have supported the research and wishes to express their gratitude to the editor and viewer for their valuable comments which have significant improved the quality of this research.

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