Proceedings of Science and Mathematics
Faculty of science, universiti teknologi malaysia
https://science.utm.my/procscimath/
Volume 26 (2024) 94-105

The Deep Enhanced Power Graph of Some Nonabelian Metabelian Group

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Abstract
This paper focuses on the deep enhanced power graph associated for nonabelian metabelian group of order at most 24. A group G is metabelian if and only if there exists an abelian normal subgroup A such that the quotient group $G/_A$ is abelian. For a finite group G, the deep enhanced power graph is defined as a simple undirected graph whose vertices represent all elements of G except for the nontrivial central element and two distinct vertices are adjacent if they belong to the same cyclic subgroup. This research starts by exploring the nonabelian metabelian group and the construction of the deep enhanced power graph using the definition and related results from past researcher in order to provide a general presentation of its structure thus providing deeper insight on their properties and characteristics.
Keywords: Nonabelian metabelian groups, Deep enhanced power graph, Dihedral group, cyclic group

Introduction

Geometric group theory provides a connection between group theory and graph theory, which are both significant areas of mathematics. The vertex adjacency of the graph is employed to assess the connections between the elements or subgroups inside the group (Bello, 2020). The graph associated with the group is a widely recognised and very practical combination, with numerous valuable applications (Li, 2021). After the researchers introduced the graphs of groups, it is crucial to ascertain the overall representation. The broad presentation facilitates the connection between the specified graphs and certain categories of finite groups, aiding researchers in efficiently acquiring the qualities. A group G is metabelian if and only if there exists an abelian normal subgroup A such that the quotient group G/A is also abelian (Wisneskey, 2005). It is intriguing to examine the portrayal of the nonabelian metabelian group, as it holds significant importance in the field of group theory (Samaila, 2013).

Multiple researches have been undertaken to examine the algebraic characteristics of the group by utilising the corresponding graph. An interesting graph is the enhanced power graph, positioned between a commuting graph and a power graph (Aalipour, 2016). Let G be a finite group. The enhanced power graph, $\Gamma_{DEP}(G)$, is defined as the simple undirected graph with vertex set G and two distinct vertices, x and y are adjacent if they belongs to the same cyclic subgroup of G (Dalal and Kumar, 2021).The features and graph invariants of this described graph have been extensively researched in recent years (see also (Prasad Panda and Kumar, 2020), (Dalal and Kumar, 2021), (Zahirović, 2020)). In addition, this graph is sometimes referred to a cyclic graph (Kelarev, 2022) and further information regarding its features may be found in (Ma, 2022). A "deleted enhanced power graph" is created by eliminating the identity element of G from its original vertices, resulting in a new subgraph, $\Gamma_{DEP}(G)$, (see to (Constanzo, 2021)). In addition, the concept of selecting a distinct set of vertices was employed to establish a subgraph within the commuting graph, focusing on non-central elements of G as the vertex set (Najmuddin, 2021). Thus, the new subgraph of $\Gamma_{DEP}(G)$ can be distinguished by examining the distinct set of vertices.

This paper aims to determine the deep enhanced power graph associated with some of the nonabelian metabelian group up to order 24. By constructing and presenting their general presentation, the characteristic and pattern of the graph is simplified.

Preliminaries and methods

This section presents some related information and the methods of constructing the deep enhanced power graph for some nonabelian metabelian group as listed in Table 1. Noted that all metabelian groups have been determined by (Fatimah, 2010)

Table 1. Some nonabelian metabelian group of order at most 24

No.	Group	$\lvert G \rvert$	Group Presentation
1.	D_3	6	$\langle a,b \mid a^3=b^2=e, bab=a^{-1}\rangle$
2.	D_4	8	$\langle a,b \mid a^4=b^2=e, bab=a^{-1}\rangle$
3.	D_5	10	$\langle a,b \mid a^5=b^2=e, bab=a^{-1}\rangle$
4.	$\mathbf{Z}_3 \rtimes \mathbf{Z}_4$	12	$\langle a,b \mid a^4=b^3=e, aba=a\rangle$
5.	D_6	12	$\langle a,b \mid a^6=b^2=e, bab=a^{-1}\rangle$
6.	D_7	14	$\langle a,b \mid a^7=b^2=e, bab=a^{-1}\rangle$
7.	D_8	16	$\langle a,b \mid a^8=b^2=e, bab=a^{-1}\rangle$
8.	D_9	18	$\langle a,b \mid a^9=b^2=e, bab=a^{-1}\rangle$
9.	D_{10}	20	$\langle a,b \mid a^{10}=b^2=e, bab=a^{-1}\rangle$
10.	$\mathbf{Z}_5$ ã $\mathbf{Z}_4$	20	$\langle a,b \mid a^5=b^4=e, aba=b\rangle$
11.	$\mathbf{Z}_4$ ã $\mathbf{Z}_5$	20	$\langle a,b \mid a^4=b^5=e, ba=a\rangle$
12.	$\mathbf{Z}_7$ ã $\mathbf{Z}_3$	21	$\langle a,b \mid a^3=b^7=e, ba=ab^2\rangle$

13.	D_{11}	22	$\langle a,b \mid a^{11}=b^2=e, bab=a^{-1}\rangle$
14.	D_{12}	24	$\langle a,b \mid a^{11}=b^2=e, bab=a^{-1}\rangle$
15.	$\mathbf{Z}_3$ ã $\mathbf{Z}_8$	24	$\langle a,b \mid a^4=b^6=e, bab=a\rangle$

Definition 1. [11] A group G is metabelian if there exist a normal subgroup A such that both A and G/A are abelian.

Definition 2. [6] For $n \in$ ¢ and $n \geq 3$, the dihedral group of order $2n$, D_n is defined by

$$D_n = \left\langle a,b \middle| a^n = b^2 = (ab)^2 = e, ab = b^{-1}a \right\rangle.$$

Definition 3. [15] The cyclicizer of the element x in G , $Cyc_G(x)$, is expressed by

$$Cyc_G(x) = \left\{ y \in G \middle| \langle x,y \rangle \text{ is cyclic } \right\}.$$

Definition 4. [8] Let G be a finite group and X be all elements in G except the non-trivial central element of G. Deep enhanced power graph, $\Gamma_{DEP}(G,X)$, is a simple undirected graph that has the elements of X as its vertices and two distinct vertices x and y are adjacent if and only if $\langle x,y \rangle$ is a proper cyclic subgroup of G.

Theorem 1. [8] Let D_n be the dihedral group of order $2n$ where $n \geq 3$. Then,

$$\Gamma_{DEP} = \begin{cases} K_1 + (K_{n-1} \cup nK_1) & \text{if } n \text{ is odd} \\ K_1 + (K_{n-2} \cup nK_1) & \text{if } n \text{ is even} \end{cases}$$

The method begins by constructing the deep enhanced power graph for all dihedral group using Theorem 1 [8]. Next, all semi direct product between cyclic groups in nonabelian metabelian group of order at most 24 listed in Table 1 will be constructed using Definition 4 [8] which includes all elements except the nontrivial central element a^n, with vertices connected if they belong to the same cyclic subgroup.

Results and discussions

This section begins with the construction of the deep enhanced graph for all dihedral groups listed in Table 1 by using Theorem 1.

Proposition 1 Let D_n be the dihedral group of order $2n$ where $3 \leq n \leq 12$. Then, the deep enhanced power graph of D_n are as follows:

$$\Gamma_{DEP}(D_n) = \begin{cases} K_1 + (K_2 \cup 3K_1) & ; \quad n = 3 \\ K_1 + (K_2 \cup 4K_1) & ; \quad n = 4 \\ K_1 + (K_4 \cup 5K_1) & ; \quad n = 5 \\ K_1 + (K_4 \cup 6K_1) & ; \quad n = 6 \\ K_1 + (K_6 \cup 7K_1) & ; \quad n = 7 \\ K_1 + (K_6 \cup 8K_1) & ; \quad n = 8 \\ K_1 + (K_8 \cup 9K_1) & ; \quad n = 9 \\ K_1 + (K_8 \cup 10K_1) & ; \quad n = 10 \\ K_1 + (K_{10} \cup 11K_1) & ; \quad n = 11 \\ K_1 + (K_{10} \cup 12K_1) & ; \quad n = 12 \end{cases}$$

Proof by Theorem 1, where n is replace by 3 until 12, the results hold. Next, the deep enhanced power graph is constructed for all semi direct product between cyclic groups listed in Table 1 using Definition 4.

Proposition 2. Let ¢ $_3$ã ¢ $_4$ be the nonabelian metabelian group of order 12. Then the deep enhanced power graph of ¢ $_3$ã ¢ $_4$ is

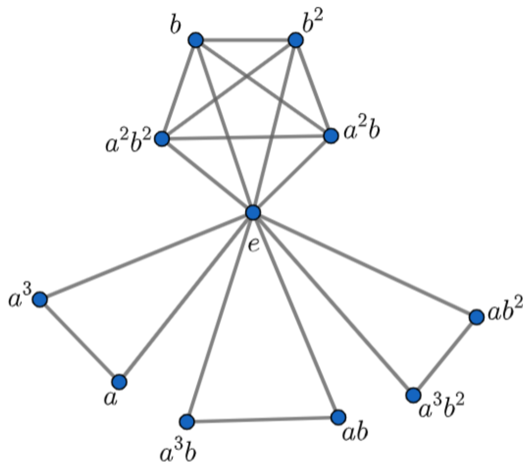

Figure 1. Deep Enhanced Power Graph of ¢ $_3$ã ¢ $_4$

Proof Given the group presentation of $¢_3 ã ¢_4$ as $\langle a,b \mid a^4 = b^3 = e, bab = a \rangle$ and the elements in this group are $\{e, a^2, b, a^3b^2, a, a^3, ab, a^3b, b^2, a^2b^2, a^2b, ab^2\}$ where the centre of the element of the group are $\{e, a^2\}$. Therefore, the nontrivial central element is $\{a^2\}$. Hence, by Definition 4, $X = \{e, b, a^3b^2, a, a^3, ab, a^3b, b^2, a^2b^2, a^2b, ab^2\}$. Now, $\Gamma_{DEP}(¢_3 ã ¢_4, X)$ is constructed by obtaining the information about the adjacency of all element x in $V(\Gamma_{DEP}(¢_3 ã ¢_4, X))$ as given in Table 2. Here, Definition 3 need to be used.

Table 2: The adjacency of element x in $\Gamma_{DEP}(¢_3 ã ¢_4, X)$

x	$Cyc_{(¢_3 ã ¢_4, X)}(x)$	The adjacency of x
e	$\{e, b, a^3b^2, a, a^3, ab, a^3b, b^2, a^2b^2, a^2b, ab^2\}$	$b, a^3b^2, a, a^3, ab, a^3b, b^2, a^2b^2, a^2b, ab^2$
a	$\{e, a, a^3\}$	e, a^3
a^3	$\{e, a, a^3\}$	e, a
b	$\{e, b, b^2, a^2b^2, a^2b\}$	e, b^2, a^2b^2, a^2b
b^2	$\{e, b, b^2, a^2b^2, a^2b\}$	e, b, a^2b^2, a^2b
ab	$\{e, ab, a^3b\}$	e, a^3b
a^2b	$\{e, b, b^2, a^2b^2, a^2b\}$	e, b^2, a^2b^2, b
a^3b	$\{e, ab, a^3b\}$	e, ab
ab^2	$\{e, a^3b^2, ab^2\}$	e, a^3b^2
a^2b^2	$\{e, b, b^2, a^2b^2, a^2b\}$	e, b^2, b, a^2b
a^3b^2	$\{e, a^3b^2, ab^2\}$	e, ab^2

Proposition 3. Let $\mathbf{Z}_5 ã \mathbf{Z}_4$ be the nonabelian metabelian group of order 20. Then, the deep enhanced power graph of $\mathbf{Z}_5 ã \mathbf{Z}_4$ is

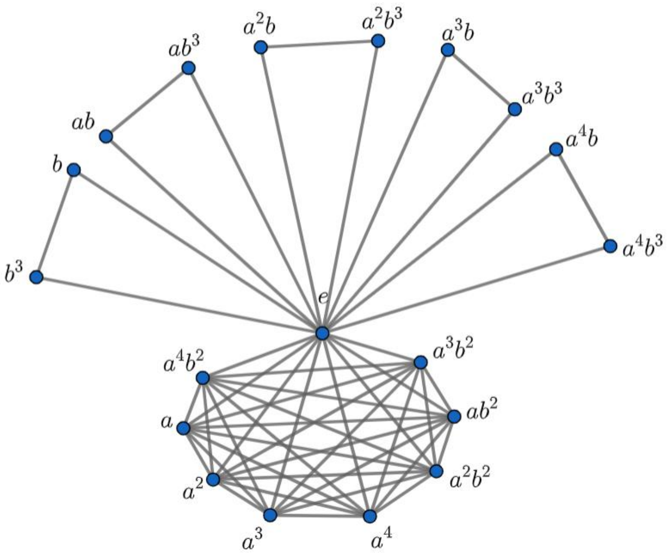

Figure 2. Deep Enhanced Power Graph of $\mathbf{Z}_5 \rtimes \mathbf{Z}_4$

Proof Given the group presentation of $\mathbf{Z}_5 \rtimes \mathbf{Z}_4$ as $\langle a, b \mid a^5 = b^4 = e, aba = b \rangle$ and the elements in this group are $\{e, a, b, a^2, a^3, a^4, b^2, b^3, ab, a^2b, a^3b, a^4b, ab^2, a^2b^2, a^3b^2, a^4b^2, ab^3, a^2b^3, a^3b^3, a^4b^3\}$ where the centre of the element of the group are $\{e, b^2\}$. Therefore, the nontrivial central element is $\{b^2\}$. Hence, by Definition 4, $X = \{e, a, b, a^2, a^3, a^4, b^3, ab, a^2b, a^3b, a^4b, ab^2, a^2b^2, a^3b^2, a^4b^2, ab^3, a^2b^3, a^3b^3, a^4b^3\}$. Now, $\Gamma_{DEP}(\mathbf{Z}_5 \rtimes \mathbf{Z}_4, X)$ is constructed by obtaining the information about the adjacency of all element x in $V(\Gamma_{DEP}(\mathbf{Z}_5 \rtimes \mathbf{Z}_4, X))$ as given in Table 3.

Table 3: The adjacency of element x in $\Gamma_{DEP}(\mathbf{Z}_5 \rtimes \mathbf{Z}_4, X)$

x	$Cyc_{(\mathbf{Z}_5 \rtimes \mathbf{Z}_4, X)}(x)$	The adjacency of x
e	X	X
a	$\{e, a, a^2, a^3, a^4\}$	e, a^2, a^3, a^4
b	$\{e, b, b^3\}$	e, b^3
a^2	$\{e, a, a^2, a^3, a^4\}$	e, a, a^3, a^4
a^3	$\{e, a, a^2, a^3, a^4\}$	e, a, a^2, a^4
a^4	$\{e, a, a^2, a^3, a^4\}$	e, a, a^2, a^3
b^3	$\{e, b, b^3\}$	e, b
ab	$\{e, ab, ab^3\}$	e, ab^3
a^2b	$\{e, a^2b, a^2b^3\}$	e, a^2b^3
a^3b	$\{e, a^3b, a^3b^3\}$	e, a^3b^3
a^4b	$\{e, a^4b, a^4b^3\}$	e, a^4b^3
ab^2	$\{e, a, a^2, a^3, a^4, ab^2, a^2b^2, a^3b^2, a^4b^2\}$	$e, a, a^2, a^3, a^4, a^2b^2, a^3b^2, a^4b^2$
a^2b^2	$\{e, a, a^2, a^3, a^4, ab^2, a^2b^2, a^3b^2, a^4b^2\}$	$e, a, a^2, a^3, a^4, a^3b^2, a^4b^2, ab^2$
a^3b^2	$\{e, a, a^2, a^3, a^4, ab^2, a^2b^2, a^3b^2, a^4b^2\}$	

a^4b^2	$\{e,a,a^2,a^3,a^4,ab^2,a^2b^2,a^3b^2,a^4b^2\}$	$e,a,a^2,a^3,a^4,a^2b^2,a^4b^2,ab^2$
ab^3	$\{e,ab,ab^3\}$	$e,a,a^2,a^3,a^4,a^2b^2,a^3b^2,ab^2$
a^2b^3	$\{e,a^2b,a^2b^3\}$	e,ab
a^3b^3	$\{e,a^3b,a^3b^3\}$	e,a^2b
a^4b^3	$\{e,a^4b,a^4b^3\}$	e,a^4b

Proposition 4. Let $\mathbf{Z}_4$ ã $\mathbf{Z}_5$ be the nonabelian metabelian group of order 20. Then, the deep enhanced power graph for $\mathbf{Z}_4$ ã $\mathbf{Z}_5$ is

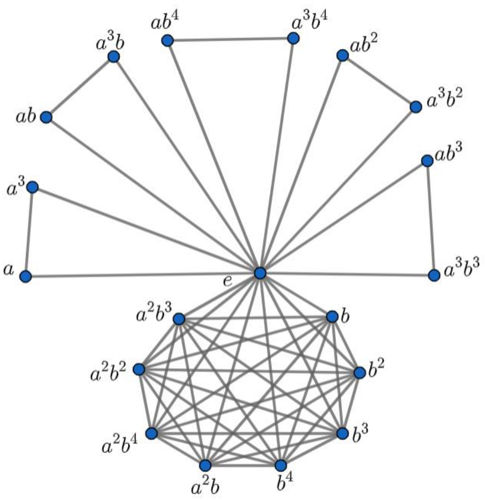

Figure 3. Deep Enhanced Power Graph of $\mathbf{Z}_4$ ã $\mathbf{Z}_5$

Proof Given the group presentation of $\mathbf{Z}_4$ ã $\mathbf{Z}_5$ as $\langle a,b \mid a^4=b^5=e, bab=a\rangle$ and the elements in this group are $\mathbf{Z}_4$ ã $\mathbf{Z}_5 = \{e,a,b,a^2,a^3,ab,a^2b,a^3b,b^3,ab^3,a^2b^3,a^3b^3,b^2,ab^2,a^2b^2,a^3b^2,b^4,ab^4,$ $a^2b^4,a^3b^4\}$ where the centre of the element of the group are $\{e,a^2\}$. Therefore, the nontrivial central element is $\{a^2\}$. Hence, by Definition 4, $X=\{e,a,b,\ a^3,ab,a^2b,a^3b,b^3,ab^3,a^2b^3,a^3b^3,b^2,\ ab^2,$ $a^2b^2,\ a^3b^2,\ b^4,ab^4,a^2b^4,a^3b^4\}$. Now, the $\Gamma_{DEP}(\mathbf{Z}_4$ ã $\mathbf{Z}_5, X)$ is constructed by obtaining the information about the adjacency of all element x in $V(\Gamma_{DEP}(\mathbf{Z}_4$ ã $\mathbf{Z}_5, X))$ as given in Table 4.

Table 4: The adjacency of element x in $\Gamma_{DEP}(\mathbf{Z}_4$ ã $\mathbf{Z}_5, X)$

x	$Cyc_{(\mathbf{Z}_4 ã \mathbf{Z}_5, X)}(x)$	The adjacency of x
e	X	X
a	$\{e,a,a^3\}$	e,a^3

b	$\{e,b,b^2,b^3,b^4\}$	$e,b^2,b^3,b^4,a^2b,a^2b^4,a^2b^2,a^2b^3$
b^2	$\{e,b,b^2,b^3,b^4\}$	$e,b,b^3,b^4,a^2b,a^2b^4,a^2b^2,a^2b^3$
a^3	$\{e,a,a^3\}$	e,a
b^4	$\{e,b,b^2,b^3,b^4\}$	$e,b,b^2,b^3,a^2b,a^2b^4,a^2b^2,a^2b^3$
b^3	$\{e,b,b^2,b^3,b^4\}$	$e,b,b^2,b^4,a^2b,a^2b^4,a^2b^2,a^2b^3$
ab	$\{e,ab,a^3b\}$	e,a^3b
ab^2	$\{e,ab^2,a^3b^2\}$	e,a^3b^2
a^3b	$\{e,ab,a^3b\}$	e,ab
ab^4	$\{e,ab^4,a^3b^4\}$	e,a^3b^4
a^2b	$\{e,b,b^2,b^3,b^4,a^2b,a^2b^2,a^2b^3,a^2b^4\}$	$e,b,b^2,b^3,b^4,a^2b^2,a^2b^3,a^2b^4$
a^2b^2	$\{e,b,b^2,b^3,b^4,a^2b,a^2b^2,a^2b^3,a^2b^4\}$	$e,b,b^2,b^3,b^4,a^2b,a^2b^3,a^2b^4$
a^2b^3	$\{e,b,b^2,b^3,b^4,a^2b,a^2b^2,a^2b^3,a^2b^4\}$	$e,b,b^2,b^3,b^4,a^2b,a^2b^2,a^2b^4$
a^2b^4	$\{e,b,b^2,b^3,b^4,a^2b,a^2b^2,a^2b^3,a^2b^4\}$	$e,b,b^2,b^3,b^4,a^2b,a^2b^2,a^2b^3$
ab^3	$\{e,a^3b^3,ab^3\}$	e,a^3b^3
a^3b^2	$\{e,ab^2,a^3b^2\}$	e,ab^2
a^3b^3	$\{e,a^3b^3,ab^3\}$	e,ab^3
a^3b^4	$\{e,ab^4,a^3b^4\}$	e,ab^4

Proposition 5. Let $\mathbf{Z}_7$ ã $\mathbf{Z}_3$ be a nonabelian metabelian group of order 21. Then, the deep enhanced power graph of $\mathbf{Z}_7$ ã $\mathbf{Z}_3$ is

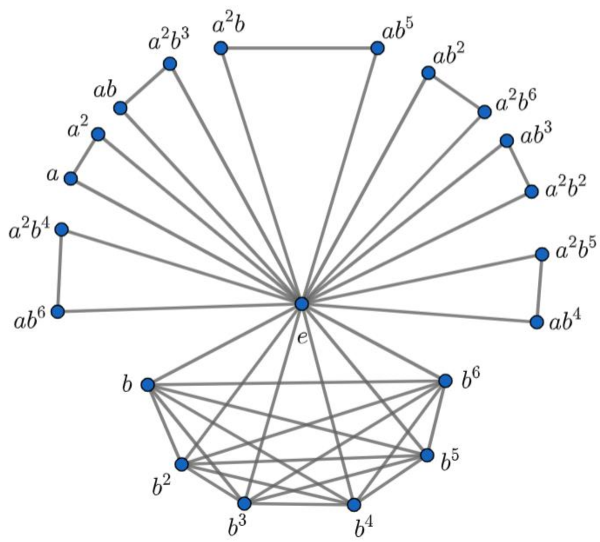

Figure 4. Deep Enhanced Power Graph of $\mathbf{Z}_7$ ã $\mathbf{Z}_3$

Proof Given the group presentation of $\mathbf{Z}_7 \tilde{a}\ \mathbf{Z}_3$ as $\langle a,b \mid a^3 = b^7 = e, ba = ab^2 \rangle$ and the elements in this group are $\mathbf{Z}_7 \tilde{a}\ \mathbf{Z}_3 = \{e, a, a^2, b, b^2, b^3, b^4, b^5, b^6, ab, ab^2, ab^3, ab^4, ab^5, ab^6,\ a^2b, a^2b^2, a^2b^3,$ $a^2b^4,\ a^2b^5, a^2b^6\}$ where the centre of the element of the group are $\{e\}$. Therefore, there are no nontrivial central element. Hence, by Definition 4, $X = \{e, a, a^2, b, b^2, b^3, b^4, b^5, b^6, ab,\ ab^2, ab^3,$ $ab^4, ab^5, ab^6,\ a^2b^2, a^3b^2,\ b^4, ab^4, a^2b^4, a^3b^4\}$. Now, $\Gamma_{DEP}(\mathbf{Z}_7 \tilde{a}\ \mathbf{Z}_3, X)$ is constructed by obtaining the information about the adjacency of all element x in $V(\Gamma_{DEP}(\mathbf{Z}_7 \tilde{a}\ \mathbf{Z}_3, X))$ as given in Table 5.

Table 5: The adjacency of element x in $\Gamma_{DEP}(\mathbf{Z}_7 \tilde{a}\ \mathbf{Z}_3, X)$

x	$Cyc_{(\mathbf{Z}_7 \tilde{a}\ \mathbf{Z}_3)}(x)$	The adjacency of x
e	X	X
b	$\{e, b, b^2, b^3, b^4, b^5, b^6\}$	$e, b^2, b^3, b^4, b^5, b^6$
b^2	$\{e, b, b^2, b^3, b^4, b^5, b^6\}$	e, b, b^3, b^4, b^5, b^6
b^3	$\{e, b, b^2, b^3, b^4, b^5, b^6\}$	e, b, b^2, b^4, b^5, b^6
b^4	$\{e, b, b^2, b^3, b^4, b^5, b^6\}$	e, b, b^2, b^3, b^5, b^6
b^5	$\{e, b, b^2, b^3, b^4, b^5, b^6\}$	e, b, b^2, b^3, b^4, b^6
b^6	$\{e, b, b^2, b^3, b^4, b^5, b^6\}$	e, b, b^2, b^3, b^4, b^5
ab^6	$\{e, ab^6, a^2b^4\}$	e, a^2b^4
a^2b^4	$\{e, ab^6, a^2b^4\}$	e, ab^6
a	$\{e, a, a^2\}$	e, a^2
a^2	$\{e, a, a^2\}$	e, a
ab	$\{e, ab, a^2b^3\}$	e, a^2b^3
a^2b^3	$\{e, ab, a^2b^3\}$	e, ab
a^2b	$\{e, a^2b, ab^5\}$	e, ab^5
ab^5	$\{e, a^2b, ab^5\}$	e, a^2b
ab^2	$\{e, a^2b^6, ab^2\}$	e, a^2b^6
a^2b^6	$\{e, a^2b^6, ab^2\}$	e, ab^2
ab^3	$\{e, ab^3, a^2b^2\}$	e, a^2b^2
a^2b^2	$\{e, ab^3, a^2b^2\}$	e, ab^3
a^2b^5	$\{e, ab^4, a^2b^5\}$	e, ab^4
ab^4	$\{e, ab^4, a^2b^5\}$	e, a^2b^5

Proposition 6. Let $\mathbf{Z}_3 \tilde{a}\ \mathbf{Z}_8$ be a nonabelian metabelian group of order 24. Then, the deep enhanced power graph of $\mathbf{Z}_3 \tilde{a}\ \mathbf{Z}_8$ is

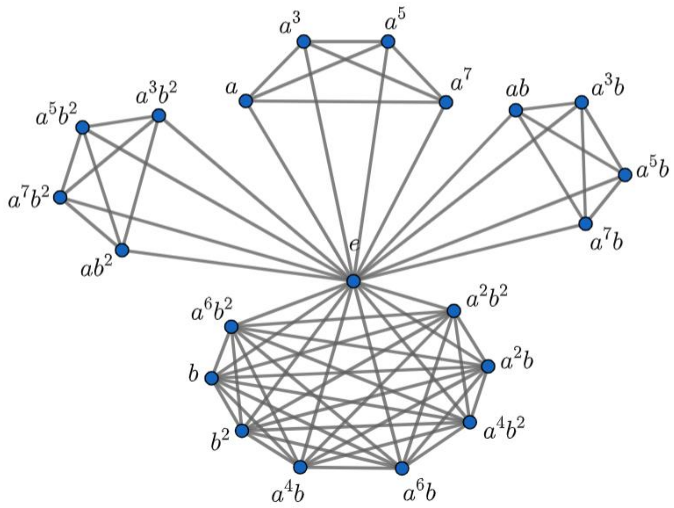

Figure 5. Deep Enhanced Power Graph of $\mathbf{Z}_3$ã $\mathbf{Z}_8$

Proof Given the group presentation of $\mathbf{Z}_3$ã $\mathbf{Z}_8$ as $\langle a,b,| a^8 = b^3 = e, bab = a\rangle$ and the elements in this group are $\mathbf{Z}_3$ã $\mathbf{Z}_8 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, b^2, ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2, a^7b^2\}$ where the centre of the element of the group are $\{e, a^2, a^4, a^6\}$. Therefore, the nontrivial central element is $\{a^2, a^4, a^6\}$. Hence, by Definition 4, $X = \{e, a, a^3, a^5, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, b^2, ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2, a^7b^2\}$. Now, $\Gamma_{DEP}(\mathbf{Z}_3$ ã $\mathbf{Z}_8, X)$ is constructed by obtaining the information about the adjacency of all element x in $V(\Gamma_{DEP}(\mathbf{Z}_3$ã $\mathbf{Z}_8, X))$ as given in Table 6.

Table 6: The adjacency of element x in $\Gamma_{DEP}(\mathbf{Z}_3$ã $\mathbf{Z}_8, X)$

x	$Cyc_{(\mathbf{Z}_3ã\,\mathbf{Z}_8, X)}(x)$	The adjacency of x
e	X	X
b	$\{e, b, b^2\}$	$e, b^2, a^4b, a^2b, a^6b, a^2b^2, a^4b^2, a^6b^2$
b^2	$\{e, b, b^2\}$	$e, b, a^4b, a^2b, a^6b, a^2b^2, a^4b^2, a^6b^2$
a^6b^2	$\{e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2\}$	$e, b, b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2$
a^4b	$\{e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2\}$	$e, b, b^2, a^6b^2, a^6b, a^4b^2, a^2b, a^2b^2$
a^6b	$\{e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2\}$	$e, b, b^2, a^6b^2, a^4b, a^4b^2, a^2b, a^2b^2$
a^4b^2	$\{e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2\}$	$e, b, b^2, a^6b^2, a^4b, a^6b, a^2b, a^2b^2$
a^2b	$\{e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2\}$	$e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b^2$
a^2b^2	$\{e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b, a^2b^2\}$	$e, b, b^2, a^6b^2, a^4b, a^6b, a^4b^2, a^2b$

ab^2	$\{e, ab^2, a^7b^2, a^5b^2, a^3b^2\}$	$e, a^7b^2, a^5b^2, a^3b^2$
a^7b^2	$\{e, ab^2, a^7b^2, a^5b^2, a^3b^2\}$	e, ab^2, a^5b^2, a^3b^2
a^5b^2	$\{e, ab^2, a^7b^2, a^5b^2, a^3b^2\}$	e, ab^2, a^7b^2, a^3b^2
a^3b^2	$\{e, ab^2, a^7b^2, a^5b^2, a^3b^2\}$	e, ab^2, a^7b^2, a^5b^2
a	$\{e, a, a^3, a^5, a^7\}$	e, a^3, a^5, a^7
a^3	$\{e, a, a^3, a^5, a^7\}$	e, a, a^5, a^7
a^5	$\{e, a, a^3, a^5, a^7\}$	e, a, a^3, a^7
a^7	$\{e, a, a^3, a^5, a^7\}$	e, a, a^3, a^5
ab	$\{e, ab, a^3b, a^5b, a^7b\}$	e, a^3b, a^5b, a^7b
a^3b	$\{e, ab, a^3b, a^5b, a^7b\}$	e, ab, a^5b, a^7b
a^5b	$\{e, ab, a^3b, a^5b, a^7b\}$	e, ab, a^3b, a^7b
a^7b	$\{e, ab, a^3b, a^5b, a^7b\}$	e, ab, a^3b, a^5b

Conclusion

In this paper, the deep enhanced power graph for all dihedral groups and semi direct product between cyclic groups listed which are also nonabelian metabelian group of order at most 24 are determined and constructed. Since this research focuses only on some groups listed in the nonabelian metabelian group of order at most 24, the deep enhanced power graph can be extended by constructing the rest of the group considered in the nonabelian metabelian group of order at most 24 to study the characteristic of the graph, its similarities and difference. This range offers valuable information about the structural characteristics and connections of the graph. Our research enhances the comprehension of the spectral properties of graphs linked to finite groups, hence creating opportunities for more investigations in both theoretical and applied graph theory.

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