Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 23, 2024, page 99-107

# Numerical Solutions of 2D Poisson Equation Using Finite Element Method and Finite Difference Method

**Siti Nuraisah Roslan, Yeak Su Hoe**
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
Corresponding author: s.h.yeak@utm.my

**Abstract**
The Poisson equation frequently emerges in many fields of science and engineering. Given the rarity of exact solutions, numerical approaches like the Finite Difference Method (FDM) and Finite Element Method (FEM) are crucial. This paper provides a comprehensive comparison of FDM and FEM in solving the 2D Poisson equation for heat transfer problems. Both methods are implemented in Python, focusing on their accuracy, computational efficiency, and suitability for different geometries and boundary conditions. Results found that FEM shows more accurate adaptability while FDM shows efficiency in handling simpler and structured problem programming but faces less accuracy than FEM. The choice of method should be tailored to the specific problem requirements. The development of Python code increases the practical applicability of this method, facilitating further advances in numerical analysis and heat transfer simulation.
**Keywords:** 2D Poisson equation; Finite Difference Method (FDM); Finite Element Method (FEM); numerical analysis

## 1. Introduction

Heat transfer is a fundamental process involving the exchange of energy between bodies due to temperature differences [1]. This exchange can occur through conduction, convection, and radiation. Although the mechanisms and laws governing the three modes of heat transfer are quite different, all three modes can occur during one process. When there is a temperature difference between two bodies, energy is transferred from the hotter object to the colder object, and this transfer only occurs in the direction of decreasing temperature. An important industrial device to enable the transfer of heat between fluids is called a heat exchanger [2]. When molecules come into contact, heat is transferred through the material, while radiation describes the transfer of energy through electromagnetic waves, such as light [3]

Partial Differential Equations (PDEs) play a crucial role in describing various natural phenomena, including heat transfer. Industries such as energy, transportation and manufacturing rely heavily on heat transfer professionals to optimize processes and systems [4]. Historically, researchers have tackled PDE problems through a combination of experimental, analytical, and numerical approaches, by making comparisons between results and exact values while carefully evaluating errors in approximate solutions. Numerical analysis is a branch of mathematics that deals with creating efficient methods for obtaining numerical solutions to difficult mathematical problems. Most of the mathematical problems that arise in science and engineering are very hard and sometimes impossible to solve exactly [5].

Numerical analysis is essential for solving complex mathematical problems in science and engineering. An important application of numerical techniques is in dealing with PDE problems related to heat transfer. The Finite Difference Method (FDM) and the Finite Element Method (FEM) are two commonly used numerical approaches for heat transfer calculations. These methods are often compared by explaining their fundamentals and applying them to common heat transfer problems. In general, it is noted that both methods are clearly superior, and in many cases, the choice depends more on the state of the problem to be solved. FEM has emerged as the most important numerical technique, known for its adaptability in addressing engineering and mathematical physics challenges [6].

This research focuses on developing numerical solutions for the two-dimensional (2D) Poisson equation, a key element in characterizing heat flow and distribution in various applications. Specifically, this study aims to compare FEM and FDM in dealing with the 2D Poisson equation. Python was chosen as the computational tool because of its robust capabilities, offering reliable software tools to handle the complexity of the problem. The research aims to provide valuable insights into the strengths and limitations of both methods in heat transfer simulation, thereby improving the understanding and application of numerical techniques in solving complex 2D Poisson equations.

## 2. Literature Review

### 2.1 Heat Transfer

High precision in manufacturing requires accurate thermal modelling, as highlighted by [7]. Surface roughness affects contact heat transfer by restricting heat flow. Interfacial heat transfer between solids in contact is affected by temperature, pressure, surface texture, and material properties [8]. Modelling measures temperature distribution and heat flow, with jet escape enhancing local transfer [9]. Although heat transfer techniques can reduce thermal resistance and enable energy recovery, they often cause pressure drop. Heat conduction depends on a temperature gradient, flowing from higher temperatures to lower temperatures [10].

### 2.2 2D Steady-State Heat Conduction

Heat conduction involves solving the heat equation in a given domain, with boundary conditions being crucial [11]. While exact solutions are attainable for regular-shaped domains, irregular-shaped domains pose challenges, leading to the use of numerical schemes like finite difference, finite element, and finite volume methods [12]. The Laplace equation is often utilized in two-dimensional steady-state heat transfer, where the temperature distribution in a body can be obtained by solving this equation.

Consider the two-dimensional steady state heat transfer, where the Laplace equation can be utilized as:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+Q=0 \tag{1}$$

Where $T$ = temperature, $Q$ = heat source, and $k$ = constant
Efficient formulas for rectangular domains help in solving complex problems by transforming them into simpler geometries [13].

### 2.3 Poisson Equation

The Poisson equation is a common elliptical PDE, is crucial in fields like heat conduction, incompressible flows, electromagnetics, and porous media flows, often involving irregular solution domains that need accurate discretization [13]. The 2D Poisson equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-f(x,y);\, x,y\in\Omega \tag{2}$$

describes the spatial distribution of a scalar field $u(x,y)$ in a region $\Omega$ , with $f(x,y)$ representing a source or sink term. This equation applies to various domain shapes and boundary conditions, similar to the Laplace equation in heat transfer problems [14]. Numerical methods such as the finite difference method (FDM) replace continuous derivatives with discrete operators, offering intuitive solutions but struggling with complex geometries. The finite element method (FEM) is more effective for arbitrary geometries [15].

### 2.4 Finite Element Method (FEM)

FEM is a widely used numerical technique for solving engineering physics and mathematical problems governed by differential equations [16]. The FEM process involves identifying the governing PDE, transforming it into a weak form, and solving the finite element equation. A notable application is the modeling of the static performance of aerostatic thrust bearings. [17] used FEM for fluid-structure interactions, analyzed stress distributions and built 2D models of thrust plates, demonstrating the versatility of FEM in addressing complex engineering challenges.

### 2.5 Finite Difference Method (FDM)

FDM is a versatile numerical technique widely used to solve differential equations in engineering. [18] state that the Taylor series expansion for an irregular grid around a point can be applied to any sufficiently differentiable function, leading to a set of linear equations. By expressing this for each node in the mesh, derive the set of linear equations,

$$[A]\{Df\} - \{f\} = \{0\} \tag{3}$$

where,

$$[A] = \begin{bmatrix} h_1 & k_1 & \frac{h_1^2}{2} & \frac{k_1^2}{2} & h_1k_1 \\ h_2 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_m & k_m & \frac{h_m^2}{2} & \frac{k_m^2}{2} & h_mk_m \end{bmatrix} \tag{4}$$

$$\{f\}^T = \{f_1 - f_0, f_2 - f_0, \dots, f_m - f_0\} \tag{5}$$

where the five unknown derivatives at the point $(x_0, y_0)$ are $\{Df\}^T = \left\{\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}, \frac{\partial^2 f_0}{\partial x^2}, \frac{\partial^2 f_0}{\partial y^2}, \frac{\partial^2 f_0}{\partial x \partial x}\right\}$.

[19] noted that FDM requires a higher mesh density for accuracy due to its step boundary approximation, which poses challenges with irregular shapes. In contrast, FEM allows variable element sizes, facilitates grid adjustment and handles complex geometries better [20]. Although FDM is memory efficient and can solve simple problems with handheld calculators, its explicit formulation requires small time steps for stability, increasing computation time.

## 3. Methodology

### 3.1 Mathematical Model

In this research, the two-dimensional heat equation is used to approximate temperature distribution due to heat conduction. The focus is exclusively on conduction, a widely recognized mode of heat transfer [21]. Consider a differential control volume with a constant thickness $\tau$ in the z-direction and a heat generation rate $Q$ (W/m³). The relationship between heat rate and heat flux is crucial, as illustrated in Figure 3.1. The heat rate entering and exiting the control volume can be expressed as follows:

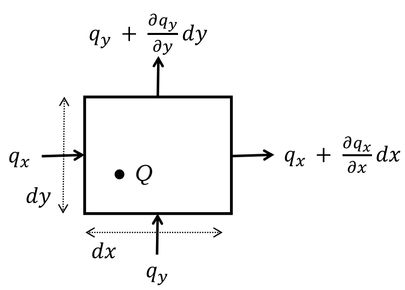


**Figure (1)**

**Figure 1** show a differential control volume for heat transfer model.

$$q_x\tau dy + q_y\tau dx + Q\tau dxdy = \left(q_x + \frac{\partial q_x}{\partial x}dx\right)\tau dy + \left(q_y + \frac{\partial q_y}{\partial y}dy\right)\tau dx \quad (6)$$

Simplifying the above equation to obtain,

$$Q = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \quad (7)$$

with the heat flux in the x and y directions given by Fourier's law:

$$q_x = -k\left(\frac{\partial T}{\partial x}\right) \text{ and } q_y = -k\left(\frac{\partial T}{\partial y}\right) \quad (8)$$

Substituting these into the heat conduction equation and get:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q = 0 \quad (9)$$

### 3.2 Finite Element Methods

Consider the strong form of the 2D steady-state heat conduction (Poisson) equation,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q = 0 \quad (10)$$

where $T$ is the temperature, $k$ is the thermal conductivity, and $Q$ is the internal heat source. Then, convert the strong form to the weak form, multiply by a test function $\phi$ and integrate over the domain $A$:

$$\iint_A \phi\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]dA + \iint_A \phi Q dA = \iint_A 0 dA = 0 \quad (11)$$

Then, express the weak form of the 2D steady-state heat conduction equation with boundary conditions, specifically including both Neumann (specified heat flux) and convective (heat transfer due to convection). So, the weak form with boundary conditions is:

$$-\int_{S_q} \phi q_0 dS - \int_{S_c} \phi h(T - T_\infty)dS - \iint_A \left(k\frac{\partial \phi}{\partial x}\frac{\partial T}{\partial x} + k\frac{\partial \phi}{\partial y}\frac{\partial T}{\partial y}\right)dA + \iint_A \phi Q dA = 0 \quad (12)$$

Discretize the domain by divide the domain $A$. Represent the temperature $T$ and test function $\phi$ in terms of shape functions $N_i$ where $T \approx \sum_{j=1}^{N} N_j T_j$ and $\phi = N_i$.

Thus, assemble the global stiffness matrix $\boldsymbol{K}$, heat transfer matrix $\boldsymbol{h}$, and the vectors $\boldsymbol{r_Q}$, $\boldsymbol{r_q}$, and $\boldsymbol{r}_\infty$.

$$\sum_{j}^{N}(h_{i,j} + k_{i,j})T_j = r_{Q,i} - r_{q,i} + r_{\infty,i} \rightarrow (\boldsymbol{h} + \boldsymbol{K})\boldsymbol{T} = \boldsymbol{r_Q} + \boldsymbol{r_q} + \boldsymbol{r}_\infty \quad (13)$$

Solve the linear system for the nodal temperatures $T_j$.

### 3.3 Finite Difference Methods

Finite difference method aims to approximate the values of the continuous function $f(x,y)$ on a set of discrete points in $(x,y)$ plane. In this method, the Poisson equation is discretized using the central difference method. Consider the Poisson equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-f(x,y) \tag{14}$$

where u is a scalar field and $f(x,y)$ is a function of a source or sink. Then, using the second-order-center-difference formulas, equation (14) can be discretized as follows:

$$\frac{u_{i+1}^{j}-2u_i^j+u_{i-1}^{j}}{\Delta x^2}+\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{\Delta y^2}=-f_i^j \tag{15}$$

For this research, it is focus on Dirichlet boundary condition where $u(x_i,y_j)$ =constant. Next, to solve for $u$ at all grid points $(i,j)$, compile the discrete equations into a matrix equation:

$$\begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,N} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,0} & A_{N,1} & \cdots & A_{N,N} \end{bmatrix}\begin{bmatrix} u_{0,0} \\ u_{1,0} \\ \vdots \\ u_{N,N} \end{bmatrix}=\begin{bmatrix} f_{0,0} \\ f_{1,0} \\ \vdots \\ f_{N,N} \end{bmatrix} \tag{16}$$

Here, $A_{i,j}$ represents the coefficients from the discretized equation. By solving this system of linear equations, the approximations for $u$ at each discrete point $(i,j)$ can be obtained.

## 4. Results and discussion

This research focuses on solving the 2D Poisson equation using FEM and FDM. These numerical methods are important for handling complex geometries and boundary conditions where analytical solutions are impractical. This method is implemented using PyCharm Edu 2022.2.2. Then, the solution is compared with the exact solution to evaluate its accuracy. This chapter details the numerical analysis and error evaluation, providing insight into the effectiveness of each method.

### 4.1 Numerical Problem

Consider the 2D Poisson equation representing a steady-state heat conduction problem:

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=-f(x,y),(x,y)\in\Omega=(0,1)\times(0,1), \tag{17}$$

where $u$ is an unknown scalar function of $x$ and $y$, and $f$ is a given function $f(x,y)$. The boundary conditions are specified as follows:

$$u(x,y)=g(x,y),(x,y)\in\delta\Omega-boundary, \qquad 0\le x\le 1,,0\le y\le 1. \tag{18}$$
$$u(0,y)=0, \qquad u(1,y)=0, \qquad u(x,0)=0, \qquad u(x,1)=0.$$

The analytical solution for this problem is given by,

$$u(x,y)=e^{-x}\cdot e^{-2y}\cdot x(1-x)\cdot y(1-y) \tag{19}$$

The right-hand side $f(x,y)$ of equation (17):

$$f(x,y)=(x^2-5x+4)(y-1)(ye^{-x-2y})+2(x-1)x(2y^2-6y+3)(e^{-x-2y}) \tag{20}$$

### 4.2 Numerical Result for Finite Element Method

The numerical solution of the 2D Poisson equation using the Finite Element Method (FEM) is represented visually in the figures. These figures provide insight into the accuracy of the approximation and the error distribution compared to the exact analytical solution.

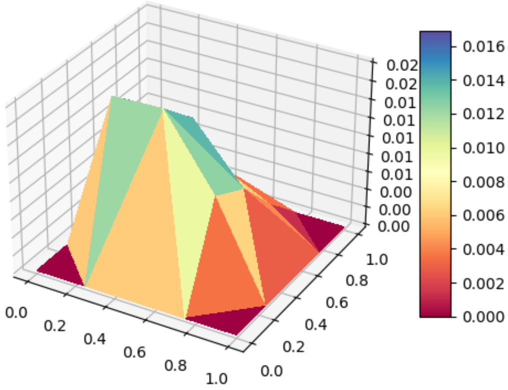


**Figure (2)**

**Figure 2** shows the numerical approximation of $u(x, y)$ obtained using FEM by solving the system of equations formed by the stiffness matrix and boundary conditions. It visually represents how $u(x, y)$ changes over the domain.

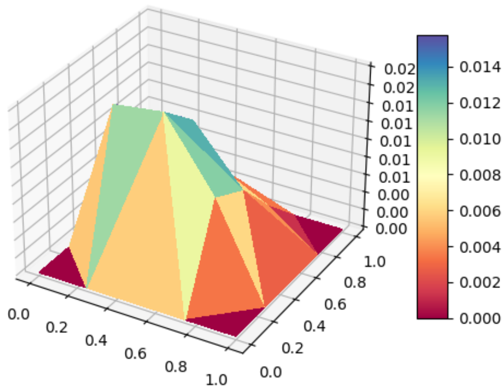


**Figure (3)**

**Figure 3** presents the exact analytical solution of $u(x, y) = x(1 - x) \cdot y(1 - y)$ for the 2D Poisson equation problem.

The error distribution between the exact and approximate solutions in FEM is given by

$$Error(x, y) = u_{numerical}(x, y) - u_{exact}(x, y)$$

This measures the difference between the FEM's numerical solution and the known exact solution at each point $(x, y)$. Analyzing this error distribution helps identify where the numerical solution deviates and guides improvements in the FEM model for better accuracy.

**4.3 Numerical Result for Finite Difference Method**

Similar to the numerical results for the Finite Element Method (FEM), the numerical solution of the 2D Poisson equation using the Finite Difference Method (FDM) is represented visually in the figure,

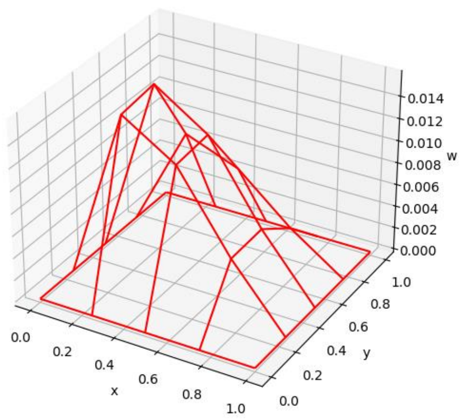


**Figure (4)**

**Figure 4** visually represents the finite difference grid and the numerical approximation process. It shows how the domain is discretized into a grid and how each grid point is updated based on the finite difference approximation.

Similar to FEM, the error distribution in the Finite Difference Method (FDM) also be calculated to helps identify where the numerical solution deviates from the exact solution, guiding improvements in the FDM model for enhanced accuracy. By examining the error distribution, it is possible to identify regions where the numerical method requires refinement, thereby improving the precision and reliability of the solutions.

**4.4 Comparison of Solutions Between FEM and FDM for a 2D Poisson Equation**

Each method for FEM and FDM discretizes the domain and approximates the solution differently, leading to varying degrees of accuracy and computational efficiency.

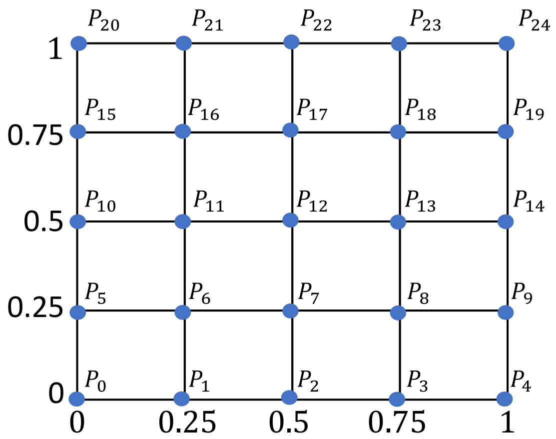


**Figure (5)**

**Figure 5** shows the layout of the nodes in a 5×5 grid for a 2D geometry problem used to solve the Poisson equation. This figure is essential for visualizing the discretization of the domain for both the Finite Element Method (FEM) and the Finite Difference Method (FDM).

The comparison between the exact and approximate solutions, along with the error, using the Finite Element Method (FEM) and Finite Difference Method (FDM) are presented in the table below:

**Table 1:** Comparison Solution of FEM and FDM

| Node | Approximate Solution for FEM | Approximate Solution for FDM | Exact Solution | Error for FEM solution | Error for FDM solution |
|---|---|---|---|---|---|
| $P_0$ | 0 | 0 | 0 | 0 | 0 |
| $P_1$ | 0 | 1.39E-17 | 0 | 0 | 1.39E-17 |
| $P_2$ | 0 | 8.33E-17 | 0 | 0 | 8.33E-17 |
| $P_3$ | 0 | -2.08E-17 | 0 | 0 | 2.08E-17 |
| $P_4$ | 0 | 0 | 0 | 0 | 0 |
| $P_5$ | 0 | 0 | 0 | 0 | 0 |
| $P_6$ | 0.018157 | 0.015483 | 0.016607 | 0.00155 | 0.001124 |
| $P_7$ | 0.018336 | 0.015928 | 0.017244 | 0.001091 | 0.001317 |
| $P_8$ | 0.010574 | 0.009238 | 0.010072 | 0.000502 | 0.000835 |
| $P_9$ | 0 | 0 | 0 | 0 | 0 |
| $P_{10}$ | 0 | 6.94E-18 | 0 | 0 | 6.94E-18 |
| $P_{11}$ | 0.014164 | 0.012446 | 0.01343 | 0.000734 | 0.000984 |
| $P_{12}$ | 0.014676 | 0.012763 | 0.013946 | 0.000731 | 0.001182 |
| $P_{13}$ | 0.008534 | 0.007384 | 0.008146 | 0.000388 | 0.000762 |
| $P_{14}$ | 0 | -4.77E-18 | 0 | 0 | 4.77E-18 |
| $P_{15}$ | 0 | -8.67E-18 | 0 | 0 | 8.67E-18 |
| $P_{16}$ | 0.006254 | 0.005589 | 0.006109 | 0.000145 | 0.000521 |
| $P_{17}$ | 0.00658 | 0.005708 | 0.006344 | 0.000236 | 0.000635 |
| $P_{18}$ | 0.003862 | 0.003292 | 0.003705 | 0.000157 | 0.000413 |
| $P_{19}$ | 0 | 0 | 0 | 0 | 0 |
| $P_{20}$ | 0 | 0 | 0 | 0 | 0 |
| $P_{21}$ | 0 | 0 | 0 | 0 | 0 |
| $P_{22}$ | 0 | 0 | 0 | 0 | 0 |
| $P_{23}$ | 0 | 0 | 0 | 0 | 0 |
| $P_{24}$ | 0 | 0 | 0 | 0 | 0 |

The exact solution values at each node represent the theoretically perfect results. The approximate FEM solution closely matches the exact solution with a much lower error compared to the FDM solution, where FDM shows a slightly larger error, indicating less accuracy in the approximation for this problem. The choice of method can significantly affect the accuracy of the solution, and FEM seems to be the better choice for problems with this particular configuration and boundary conditions.

## Conclusion

This study examines Finite Element Method (FEM) and Finite Difference Method (FDM) to solve the 2D Poisson equation using Python. The results show the successful implementation of both methods, with FEM showing higher accuracy, while FDM is efficient for computational task. This study highlights Python's effectiveness but suggests improvements in programming efficiency and diversity of examples. Future research could explore different boundary conditions (Neumann, Robin) to improve accuracy. Comparative analysis with other methods such as the Finite Volume Method (FVM) can be used to offer further insight into the efficient solution of the 2D Poisson equation.

## Acknowledgement

With deep gratitude, I thank Associate Prof. Dr. Yeak Su Hoe for his invaluable guidance and the Universiti Teknologi Malaysia for giving me the opportunity to carry out this final year project. A big thank you to my fellow undergraduates and colleagues for their support and insight. I am indebted to my family for their unwavering love, encouragement and financial support. Their sacrifice fuels my motivation. Thank you to everyone who contributed to this research. Hopefully the results will enrich the community and the field.

**References**

[1] Britannica, T. Editors of Encyclopaedia. 2023. heat. Encyclopedia Britannica.
[2] Wooldridge, M., & Luebbers, R.H. 2020. Heat transfer. AccessScience.
[3] Zahed, M. 2023. Heat Transfer applications in industry and CFD Simulation. MR CFD.
[4] Abubakar, S., Abubakar, N., & Ibrahim, M. 2019. "A STUDY ON SOME APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS IN HEAT TRANSFER".
[5] Gaur, M., Ashwin, & Chouhan, A. S. 2022. Current role of numerical analysis in Mathematics. ResearchGate.
[6] Liu, C., & Kelly, R. G. 2019. A Review of the Application of Finite Element Method (FEM) to Localized Corrosion Modeling. Corrosion.
[7] Helmig, T., & Kneer, R. 2021. A novel transient infrared-thermography based experimental method for the inverse estimation of heat transfer coefficients in rotating bearings. International Journal of Thermal Sciences.
[8] Dou, R., Ge, T., Liu, X., & Wen, Z. 2016. Effects of contact pressure, interface temperature, and surface roughness on thermal contact conductance between stainless steel surfaces under atmosphere condition. International Journal of Heat and Mass Transfer.
[9] Qiu, D., Luo, L., Zhao, Z., Wang, S., Wang, Z., & Sundén, B. 2021. On heat transfer and flow characteristics of jets impingement on a concave surface with varying pin-fin arrangements. International Journal of Thermal Sciences.
[10] Wright State University. (n.d.). chapter 3 Steady Heat Conduction (Wright State University, Ed.).
[11] Libretexts. 2022. 3.1: Introduction to boundary and initial conditions. Mathematics LibreTexts.
[12] Patel, M. R., & Pandya, J. U. 2022. Numerical study of a one and two-dimensional heat flow using finite volume. Materials Today: Proceedings, 51, 48–57.
[13] Radhakrishnan, A. 2021. A Non-Nested Multilevel Method for Meshless Solution of the Poisson Equation in Heat Transfer and Fluid Flow. arXiv.org.
[14] Holagh, S. G., Talati, F., & Shamsaiee, M. 2019. Analytical solution of steady state heat conduction equations in irregular domains with various BCs by use of Schwarz-Christoffel conformal mapping. Thermal Science and Engineering Progress.
[15] Zaman, M. A. (2022). Numerical Solution of the Poisson Equation Using Finite Difference Matrix Operators. Electronics, 11(15), 2365.
[16] Liu, C., & Kelly, R. G. 2019. A Review of the Application of Finite Element Method (FEM) to Localized Corrosion Modeling. Corrosion
[17] Gao, Q., Qi, L., Gao, S., Lu, L., Song, L., & Zhang, F. 2021. A FEM based modeling method for analyzing the static performance of aerostatic thrust bearings considering the fluid-structure interaction. Tribology International, 156.
[18] Liszka, T., and J. Orkisz. 1980. "The Finite Difference Method at Arbitrary Irregular Grids and Its Application in Applied Mechanics." Computers & Structures 11(1–2):83–95.
[19] Gomez, L. J., Dannhauer, M., Koponen, L. M., & Peterchev, A. V. 2020. Conditions for numerically accurate TMS electric field simulation. Brain Stimulation, 13(1), 157–166.
[20] Zaghloul, N. A. 1982. The application of the FEM and FDM to flow separation pattern: A comparison. Applied Mathematical Modelling, 6(1), 28–34.
[21] Xu, H., Xing, Z., Wang, F., & Cheng, Z. 2019. Review on heat conduction, heat convection, thermal radiation and phase change heat transfer of nanofluids in porous media: Fundamentals and applications. Chemical Engineering Science, 195, 462–48