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Topological Indices of the Equal Square Graph on Generalized Quaternion Group

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Abstract
Topological indices are numerical invariants of graphs that play an important role in graph theory and mathematical chemistry, particularly in describing the structural properties of molecular graphs. In this study, we investigate several topological indices associated with algebraic graphs, including the equal square graph. The graphs are derived from algebraic structures and their structural characteristics are analyzed through degree-based topological indices. Exact expressions for the considered indices are obtained by exploiting the algebraic properties of the underlying structures as well as the combinatorial features of the corresponding equal square graphs. In addition, comparative analyses are provided to highlight the influence of algebraic parameters on the values of the resulting topological indices. The results not only generalize several existing findings but also contribute to a deeper understanding of the relationship between algebraic structures and graph-theoretic invariants.
Keywords: Group; Graph; Equal Square; Topological Index.

Introduction

Graph theory is one of the branches of mathematics that continues to develop and plays a significant role in various scientific disciplines, ranging from mathematical chemistry and network theory to abstract algebra. This development has encouraged the emergence of numerous concepts and approaches aimed at understanding the structural characteristics of graphs, one of which is through the study of topological indices [1]. A topological index is a numerical invariant designed to capture structural information of a graph in the form of a number, thereby enabling quantitative analysis of the properties of that graph. Various topological indices, such as the Zagreb index, Harary index, Gutman index, and harmonic index, have been extensively investigated due to their ability to represent vertex relationships and provide mathematical interpretations of complex graph structures [2,3].

Along with the growing interest in topological index studies, research has expanded beyond general graphs to include special classes of graphs constructed from algebraic structures such as groups and rings. Coprime graphs, non-coprime graphs, power graphs, and nilpotent graphs are several examples that model relationships among elements within an algebraic structure. This perspective opens broader opportunities to connect graph theory with group theory, leading to deeper insights into structural properties and interactions among elements in algebraic systems [4].

One algebraic structure that is particularly interesting in this context is the generalized quaternion group, which is known for its non-Abelian nature and rich internal structure. From this group, various graph representations can be constructed; one of which is the equal square graph, which is a graph that represents relationships among group elements based on the equality of their squares [5]. Although studies on graphs derived from groups have grown considerably, investigations of topological indices on equal square graphs, especially those associated with generalized quaternion groups, remain relatively limited.

Therefore, this research aims to examine and determine the values of Harmonic index, ABC Index, Nirmala index, Forgotten index, and Sombor index of the equal square graph of a generalized quaternion group. Through this approach, it is expected that a more comprehensive understanding of the structural characteristics of the resulting graph can be obtained, while simultaneously enriching the interdisciplinary study between graph theory and group theory within the scope of theoretical mathematics.

Materials and methods

This research had been conducted using a theoretical and analytical approach. The study began with a comprehensive literature review concerning generalized quaternion groups, equal square graphs, and various topological indices in graph theory to establish the fundamental concepts and definitions used throughout the research. Subsequently, the structure of the equal square graph associated with the generalized quaternion group is examined by identifying its vertex set and edge set based on the defining relations of the group elements. From this structural construction, the degree of each vertex in the graph is determined through analytical derivation. After obtaining the vertex degrees, the values of the considered topological indices are then calculated in accordance with their formal mathematical definitions. This systematic procedure enables a rigorous analysis of the structural properties of the equal square graph derived from the generalized quaternion group.

Definition 1 [6] The Generalized Quaternion Group Q_{4n} is a group with representation

$$< a, b | a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1} >.$$

In this group, $a^k b = ba^{-k}$ and the order of $a^k b$ is 4.

Definition 2 [5] For a finite group G, the equal square graph of G, denoted as $ES(G)$, is defined as a graph with the vertex set $V(ES(G)) = G$, and two distinct vertices x and y are connected by an edge if and only if $x^2 = y^2$.

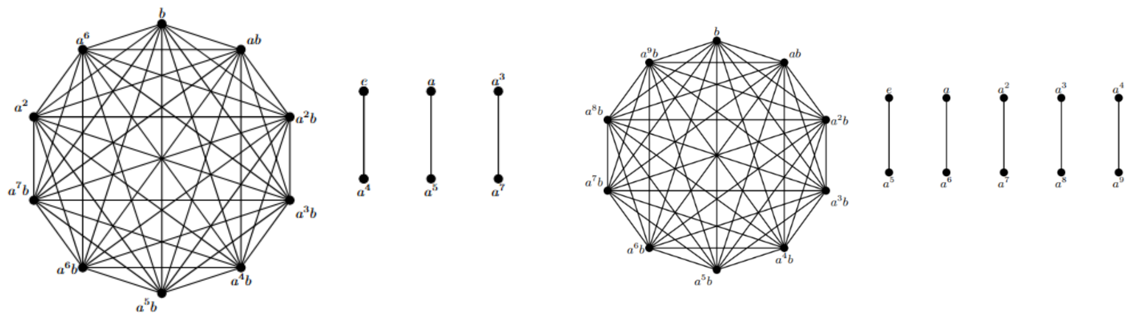
Lemma 1 [7] Given a generalized quaternion group Q_{4n}. If n is an odd integer, then

$$ES(G) = K_{2n} + nK_2.$$

Lemma 2 [7] Given a generalized quaternion group Q_{4n}. If n is an even integer, then

$$ES(G) = K_{2n+2} + (n-1)K_2.$$

As an example, Figure 1 illustrates the equal square graph of the generalized quaternion group of order 16 and 20.



(a) (b)

Figure 1 Equal Square graph representation of (a) Q_{16} and (b) Q_{20}.

Definition 3 [8] The Harmonic index of a graph G, denoted as $H(G)$ is determined by

$$H(G) = \sum_{u,v \in V(G)} \frac{2}{deg(u) + deg(v)} .$$

Definition 4 [9] The atom-bond connectivity (ABC) index of a graph, written as $ABC(G)$, which quantifies the contributions of every connected vertex pair u and v, is formulated as

$$ABC(G) = \sum_{(u,v) \in E(G)} \sqrt{\frac{deg(u) + deg(v) - 2}{deg(u)deg(v)}} ,$$

where $deg(u)$ and $deg\ (v)$ denote the degrees of vertices u and v, respectively.

Definition 5 [10] The Nirmala index of a graph G, denoted as $N(G)$ is determined by

$$N(G) = \sum_{uv \in E(G)} \sqrt{deg(u) + deg(v)},$$

where $deg\ (u)$ and $deg\ (v)$ are the degrees of adjacent vertices u and v.

Definition 6 [11] Let G be a graph. The Forgotten index of G denoted by $F(G)$, is defined as:

$$F(G) = \sum_{v \in V(G)} deg(v)^3.$$

Definition 7 [12] Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. The Sombor index of G, denoted as $SO(G)$ is defined as:

$$SO(G) = \sum_{(u,v) \in E(G)} \sqrt{deg\ deg\ (u)^{2} + deg\ deg\ (v)^{2}}.$$

Results and discussion

Referring to the previous lemmas and graphical examples, the degree of each vertex in an equal square graph of generalized quaternion group is derived as expressed in Corollary 1.

Corollary 1 Let $ES(Q_{4n})$ be the equal square graph of generalized quaternion group. Then, the degree of each vertex of $ES(Q_{4n})$ are $2n-1$ or 1 for n an odd integer, and $2n+1$ or 1 for n an even integer.

In addition, results on some topological indices of equal square graph on generalized quaternion groups are presented.

Theorem 1 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. The Harmonic index of $ES(Q_{4n})$ is

$$H(ES(Q_{4n}))=2n.$$

Proof: Case 1, for n an odd integer: From Lemma 1 and Corollary 1, there exist $2n$ vertices that are mutually adjacent to one another, so that each of these vertices has degree $2n-1$. In addition, there are $2n$ vertices that are connected to only a single vertex, forming n copies of the complete graph K_2, where each vertex has degree 1. Then, the harmonic index of $ES(Q_{4n})$ is

$$\begin{aligned}
H\big(ES(Q_{4n})\big) &= \sum_{(u,v)\in E\left(ES(Q_{4n})\right)} \frac{2}{deg(u)+deg(v)} \\
&= \sum_{(u,v)\in E_1\left(ES(Q_{4n})\right)} \frac{2}{deg(u)+deg(v)} + \sum_{(u,v)\in E_2\left(ES(Q_{4n})\right)} \frac{2}{deg(u)+deg(v)} \\
&= (2n\ 2\ )\frac{2}{(2n-1)+(2n-1)} + n\frac{2}{1+1} \\
&= \left(\frac{2n(2n-1)}{2}\right)\frac{2}{(2n-1)+(2n-1)} + n\frac{2}{1+1} \\
&= \left(\frac{4n^2-2n}{2}\right)\left(\frac{2}{4n-2}\right) + \frac{2n}{2} \\
&= 2n.
\end{aligned}$$

Case 2, for n an even integer: From Lemma 2 and Corollary 1, there exist $2n+2$ vertices that are mutually adjacent to one another, so that each of these vertices has degree $2n+1$. In addition, there are $2n-2$ vertices that are connected to only a single vertex, forming $n-1$ copies of the complete graph K_2, where each vertex has degree 1. Then, the harmonic index of $ES(Q_{4n})$ is

$$\begin{aligned}
H(ES(Q_{4n})) &= \sum_{(u,v)\in E_1(ES(Q_{4n}))} \frac{2}{deg(u)+deg(v)} + \sum_{(u,v)\in E_2(ES(Q_{4n}))} \frac{2}{deg(u)+deg(v)} \\
&= (2n+2\ 2\ )\frac{2}{(2n+1)+(2n+1)} + (n-1)\frac{2}{1+1} \\
&= \left(\frac{(2n+2)(2n+1)}{2}\right)\frac{2}{(2n+1)+(2n+1)} + (n-1)\frac{2}{1+1} \\
&= \left(\frac{4n^2+6n+2}{2}\right)\left(\frac{2}{4n+2}\right) + n - 1 \\
&= 2n.
\end{aligned}$$

Theorem 2 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an odd integer, then the ABC index of $ES(Q_{4n})$ is

$$ABC(ES(Q_{4n})) = 2n\sqrt{n-1}.$$

Proof: By applying the same approach as in Case 1 Theorem 1, the value of ABC index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$\begin{aligned}
ABC\big(ES(Q_{4n})\big) &= \sum_{(u,v)\in E\left(ES(Q_{4n})\right)} \sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}} \\
&= \sum_{(u,v)\in E_1\left(ES(Q_{4n})\right)} \sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}} + \sum_{(u,v)\in E_2\left(ES(Q_{4n})\right)} \sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}} \\
&= (2n\ 2\ )\sqrt{\frac{(2n-1)+(2n-1)-2}{(2n-1)(2n-1)}} + (n)\sqrt{\frac{1+1-2}{(1)(1)}} \\
&= \left(\frac{(2n)(2n-1)}{2}\right)\sqrt{\frac{4n-4}{4n^2-4n+1}} \\
&= 2n\sqrt{n-1}.
\end{aligned}$$

Theorem 3 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an even integer, then the ABC index of $ES(Q_{4n})$ is

$$ABC(ES(Q_{4n})) = (2n+2)\sqrt{n}.$$

Proof: By applying the same approach as in Case 2 Theorem 1, the value of ABC index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$\begin{aligned}
ABC(ES(Q_{4n})) &= \sum_{(u,v)\in E(ES(Q_{4n}))} \sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}} \\
&= \sum_{(u,v)\in E_1(ES(Q_{4n}))} \sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}} + \sum_{(u,v)\in E_2(ES(Q_{4n}))} \sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}} \\
&= (2n+2\ 2\ )\sqrt{\frac{(2n+1)+(2n+1)-2}{(2n+1)(2n+1)}} + (n-1)\sqrt{\frac{1+1-2}{(1)(1)}} \\
&= \left(\frac{(2n+2)(2n+1)}{2}\right)\sqrt{\frac{4n}{4n^2+4n+1}} \\
&= \left(\frac{(2n+2)(2n+1)}{2}\right)\left(\frac{2}{2n+1}\right)\sqrt{n} \\
&= (2n+2)\sqrt{n}.
\end{aligned}$$

Theorem 4 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an odd integer, then the Nirmala index of $ES(Q_{4n})$ is

$$N(ES(Q_{4n})) = (2n^2-n)\sqrt{4n-2} + n\sqrt{2}.$$

Proof: By applying the same approach as in Case 1 Theorem 1, the value of Nirmala index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$N(ES(Q_{4n})) = \sum_{(u,v)\in E(ES(Q_{4n}))} \sqrt{deg(u)+deg(v)}$$
$$= \sum_{(u,v)\in E_1(ES(Q_{4n}))} \sqrt{deg(u)+deg(v)} + \sum_{(u,v)\in E_2(ES(Q_{4n}))} \sqrt{deg(u)+deg(v)}$$
$$= (2n\ 2\ )\sqrt{(2n-1)+(2n-1)} + (n)\sqrt{1+1}$$
$$= \left(\frac{(2n)(2n-1)}{2}\right)\sqrt{4n-2} + n\sqrt{2}$$
$$= (2n^2-n)\sqrt{4n-2} + n\sqrt{2}.$$

Theorem 5 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an even integer, then the Nirmala index of $ES(Q_{4n})$ is

$$N(ES(Q_{4n})) = (2n^2+3n+1)\sqrt{4n+2} + (n-1)\sqrt{2}.$$

Proof: By applying the same approach as in Case 2 Theorem 1, the value of Nirmala index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$N(ES(Q_{4n})) = \sum_{(u,v)\in E(ES(Q_{4n}))} \sqrt{deg(u)+deg(v)}$$
$$= \sum_{(u,v)\in E_1(ES(Q_{4n}))} \sqrt{deg(u)+deg(v)} + \sum_{(u,v)\in E_2(ES(Q_{4n}))} \sqrt{deg(u)+deg(v)}$$
$$= (2n+2\ 2\ )\sqrt{(2n+1)+(2n+1)} + (n-1)\sqrt{1+1}$$
$$= \left(\frac{(2n+2)(2n+1)}{2}\right)\sqrt{4n+2} + (n-1)\sqrt{2}$$
$$= (2n^2+3n+1)\sqrt{4n+2} + (n-1)\sqrt{2}.$$

Theorem 6 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an odd integer, then the Forgotten index of $ES(Q_{4n})$ is

$$F(ES(Q_{4n})) = (16n^4 - 24n^3 + 12n^2).$$

Proof: By applying the same approach as in Case 1 Theorem 1, the value of Forgotten index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$F\big(ES(Q_{4n})\big) = \sum_{v\in V(ES(Q_{4n}))} deg(v)^3$$
$$= \sum_{v\in V_1(ES(Q_{4n}))} deg(v)^3 + \sum_{u\in V_2(ES(Q_{4n}))} deg(u)^3$$
$$= (2n)(2n-1)^3 + (n)(1^3+1^3)$$
$$= (2n)(8n^3-12n^2+6n-1) + 2n$$
$$= (2n)(8n^3-12n^2+6n)$$

$$= (16n^4 - 24n^3 + 12n^2).$$

Theorem 7 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an even integer, then the Forgotten index of $ES(Q_{4n})$ is

$$F(ES(Q_{4n})) = 16n^4 + 40n^3 + 36n^2 + 16n.$$

Proof: by applying the same approach as in Case 2 Theorem 1, the value of Forgotten index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$\begin{aligned} F(ES(Q_{4n})) &= \sum_{v\in V(G)} deg(v)^3 \\ &= \sum_{uv\in V(G)} deg(v)^3 + \sum_{uv\in V(G)} deg(v)^3 \\ &= (2n+2)(2n+1)^3 + (n-1)(1^3+1^3) \\ &= (2n+2)(8n^3+12n^2+6n+1) + 2n - 2 \\ &= 16n^4 + 40n^3 + 36n^2 + 16n. \end{aligned}$$

Theorem 8 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an odd integer, then the Sombor index of $ES(Q_{4n})$ is

$$SO(ES(Q_{4n})) = n\sqrt{2}(4n^2 - 2n + 2).$$

Proof: By applying the same approach as in Case 1 Theorem 1, the value of Sombor index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$\begin{aligned} SO(ES(Q_{4n})) &= \sum_{(u,v)\in E(ES(Q_{4n}))} \sqrt{deg\ deg\ (u)^{\ 2} + deg\ deg\ (v)^{\ 2}} \\ &= \sum_{(u,v)\in E_1(ES(Q_{4n}))} \sqrt{deg\ deg\ (u)^{\ 2} + deg\ deg\ (v)^{\ 2}} \\ &\quad + (n) \sum_{(u,v)\in E_2(ES(Q_{4n}))} \sqrt{deg\ deg\ (u)^{\ 2} + deg\ deg\ (v)^{\ 2}} \\ &= (2n\ 2\ )\sqrt{(2n-1)^2 + (2n-1)^2} + (n)\sqrt{1+1} \\ &= \left(\frac{(2n)(2n-1)}{2}\right)(2n-1)\sqrt{2} + n\sqrt{2} \\ &= n(2n-1)^2\sqrt{2} + n\sqrt{2} \\ &= n\sqrt{2}(4n^2 - 2n + 2). \end{aligned}$$

Theorem 9 Let $ES(Q_{4n})$ be the equal square graph of Q_{4n}. If n is an even integer, then the Sombor index of $ES(Q_{4n})$ is

$$SO(ES(Q_{4n})) = n\sqrt{2}(4n^2 - 2n + 2).$$

Proof: By applying the same approach as in Case 2 Theorem 1, the value of Sombor index of the equal square graph of generalized quaternion group Q_{4n} for n an odd integer is

$$
\begin{aligned}
SO(ES(Q_{4n})) &= \sum_{(u,v)\in E(ES(Q_{4n}))} \sqrt{deg\ deg\ (u)^{\ 2} + deg\ deg\ (v)^{\ 2}} \\
&= \sum_{(u,v)\in E_1(ES(Q_{4n}))} \sqrt{deg\ deg\ (u)^{\ 2} + deg\ deg\ (v)^{\ 2}} \\
&\quad + (n) \sum_{(u,v)\in E_2(ES(Q_{4n}))} \sqrt{deg\ deg\ (u)^{\ 2} + deg\ deg\ (v)^{\ 2}} \\
&= (2n+2\ 2\ )\sqrt{(2n+1)^2+(2n+1)^2} + (n-1)\sqrt{1+1} \\
&= \left(\frac{(2n+2)(2n+1)}{2}\right)(2n+1)\sqrt{2} + (n-1)\sqrt{2} \\
&= (n+1)(2n+1)^2\sqrt{2} + (n-1)\sqrt{2} \\
&= n\sqrt{2}(4n^2-2n+2).
\end{aligned}
$$

Conclusion

From this study, several topological index values of the equal square graph of generalized quaternion groups have been obtained, namely the Harmonic index, ABC index, Nirmala index, Forgotten index, and Sombor index. From the obtained results, it has been observed that computing the indices for the entire graph yields the same values as calculating each disjoint component separately and then adding up their index values. This is because the indices considered in this study are degree-based indices rather than distance-based indices, so the overall value is additive across disjoint graphs.

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