Proceedings of Science and Mathematics
Faculty of Science,
Universiti Teknologi Malaysia

Volume 34, 2026, page 32-37

An Analysis of Odd- and Even-Sized Cliques in the Coprime Graph of Dihedral Groups

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Abstract
The dihedral group is known to represent the rotations and reflections of regular polygons. In this paper, the coefficients of the clique polynomial of the coprime graph of some dihedral groups are analysed. The aim is to investigate the properties of the cliques of the coprime graph. The clique polynomial of the coprime graph of some dihedral groups is also determined by calculating the number of cliques according to their size. The results show that the total number of odd-sized cliques of the coprime graph of dihedral groups is equal to the total number of even-sized cliques of the graph.
Keywords: Coprime graph; dihedral group; cliques

Introduction

The coprime graph was introduced in 2014 and since then, researchers have shown interest in investigating the theoretical properties of the coprime graph [1]. For instance, the graphical representation of the coprime graph when associated to the dihedral group is determined specifically for prime cases [2]. The dihedral group is a group that represents the rotations and reflections of regular polygons [3].

Furthermore, researchers extended the study of the coprime graphs associated to the dihedral groups by determining graph invariants such as the girth, radius and diameter [4]. Both the clique numbers and chromatic numbers have also been determined for the coprime graph associated to dihedral groups [5].

Meanwhile, the cliques of a graph are used to represent close relationships such as a group of friends in a group of people [6]. The study of cliques is significant as it provides not only structural insights, but also algorithmic properties of graphs [7]. Thus, the concept of clique polynomial is needed to explain the distribution of cliques in the coprime graph of the dihedral groups.

Therefore, this paper aims to investigate the relationship between the odd-sized cliques and even-sized cliques of the coprime graph of the dihedral groups. This property is called the odd-even clique equality as it can be used to tell the structural balance within the coprime graph when associated to the dihedral groups.

Preliminaries and methods

The research starts with the definitions of the coprime graph, dihedral groups, complete graph, clique and clique polynomial which lead to several theorems for the coprime graph of dihedral groups. The clique polynomial of the coprime graph of some dihedral groups is determined. The coefficients are later analyzed to prove the equality of the number of odd-sized and even-sized of cliques.

Definition 1 [1] Coprime graph
Let G be a finite group and Γ_G^{cp} denotes the coprime graph of G. If a pair of distinct vertices in Γ_G^{cp}, u and v, where $u, v \in V(\Gamma_G^{cp})$, then the edges of Γ_G^{cp} are formed for u and v, if and only if $gcd\ (|u|, |v|)\ = 1$.

Definition 2 [3] Dihedral Groups
The dihedral group D_{2n} is the group of rotations and reflections of a regular polygon. The order of the group is $2n$. The rotations are denoted by a and the reflections denoted by b. The group presentation is,

$$D_{2n} = \langle a, b | a^n = b^2 = e, b^{-1}ab = a^{-1} \rangle, n \in N.$$

Definition 3 [8] Complete graph
Let Γ be a graph. If every vertex in Γ are adjacent to all other vertices in Γ, then Γ is called a complete graph.

Definition 4 [8] Clique
Let Γ be a graph. A clique of Γ is a subgraph of Γ which is complete. Moreover, a clique in Γ is called as a maximal clique if there is no other larger clique in Γ that contains it. The size of a maximal clique is also known as the clique number, denoted as $\omega(\Gamma)$.

Definition 5 [9] Clique polynomial
The clique polynomial of a graph Γ, is a polynomial denoted as

$$C(\Gamma; x) = \sum_{k=0}^{\omega(\Gamma)} c_k x^k,$$

where c_k represents the total number of cliques of size-k, and $\omega(\Gamma) \in N \cup 0$ is the clique number of graph Γ.

Denote $\Gamma_{D_{2n}}^{cp}$ the coprime graph of the dihedral groups. In order to determine the clique polynomial of $\Gamma_{D_{2n}}^{cp}$, the value of the clique number of the coprime graph $\omega(\Gamma_{D_{2n}}^{cp})$ is needed. Hence, Theorem 1 states the values of $\omega(\Gamma_{D_{2n}}^{cp})$ for cases of powers of 2 and the odd cases.

Theorem 1 [5]
The clique number of the coprime graph of the dihedral groups $\omega(\Gamma_{D_{2n}}^{cp})$ is given by

$$\omega(\Gamma_{D_{2n}}^{cp}) = 2m + 2\,, \qquad n = 2^k, \qquad n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}, p_i \neq 2,$$

where p_i are odd primes and m, k, k_i are natural numbers, for $1 \leq i \leq m$.

Results on odd-even clique property
In this section, the main result is obtained, followed by some propositions which support the result for specific cases. Theorem 2 is shown to cover all cases for the clique number of the coprime graph of dihedral groups.

Theorem 2
Let $\Gamma_{D_{2n}}^{cp}$ be the coprime graph of dihedral groups and $\omega(\Gamma_{D_{2n}}^{cp})$ be its clique number. If $n = 2^{k_0}p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$, where p_i are odd primes and m, k, k_i are non-negative integers, for $1 \leq i \leq m$, then $\omega(\Gamma_{D_{2n}}^{cp}) = m + 2$.

Proof. When $n = 2^{k_0}p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$, the elements of $\Gamma_{D_{2n}}^{cp}$ can be partitioned into $m + 2$ subsets based on the order of the elements. Let subsets consist elements of order 1, 2^{l_0} for some $l_0 \in N$, $p_1^{l_1}$ for some $l_1 \in N$ until $p_m^{l_m}$ for some $l_m \in N$. Since the orders of elements from each subset are coprime to each other, $\Gamma_{D_{2n}}^{cp}$ is $m + 2$ partite, which implies $\omega(\Gamma_{D_{2n}}^{cp}) = m + 2$. This covers the dihedral groups in general since any natural number n can be written in the form of prime factorization, $n = 2^{k_0}p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$. Therefore, if $n = 2^{k_0}p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$, then $\omega(\Gamma_{D_{2n}}^{cp}) = m + 2$.

The following theorem establishes the main result of this paper, which is the equality between the total number of odd-sized cliques and even-sized cliques in the coprime graph of dihedral groups.

Theorem 3
Let $\Gamma_{D_{2n}}^{cp}$ be the coprime graph of dihedral groups and $\omega = \omega(\Gamma_{D_{2n}}^{cp})$ be the clique number of the coprime graph of dihedral groups. The total number of odd-sized cliques of $\Gamma_{D_{2n}}^{cp}$ is equal to the total number of even-sized cliques of $\Gamma_{D_{2n}}^{cp}$.

$$\sum_{j=0}^{\lfloor\frac{\omega}{2}\rfloor} c_{2j} = \sum_{j=0}^{\lfloor\frac{\omega-1}{2}\rfloor} c_{2j+1},$$

where $\lfloor\cdot\rfloor$ is the floor function and c_k is the number of size-k clique in $\Gamma_{D_{2n}}^{cp}$ and k is a non-negative integer.

Proof. Based on Theorem 2, the clique number of the coprime graph of dihedral group $\omega(\Gamma_{D_{2n}}^{cp}) = m + 2$ tells us that there exist a complete multipartite subgraph of $\Gamma_{D_{2n}}^{cp}$ that is $(m + 2)$-partite. Since the order of the identity of the dihedral group $|\{e\}|$ is equal to 1, implies $\{e\}$ is connected to all the other vertices in $\Gamma_{D_{2n}}^{cp}$. Thus, the maximal clique of $\Gamma_{D_{2n}}^{cp}$ must involve $\{e\}$.

For cliques of other sizes, the cliques of the same size can be classified into two types which are cliques that include $\{e\}$ and those that do not. A repetitive pattern can be analysed such that for any size-k clique that do not include $\{e\}$, it can always be a size-$(k + 1)$ clique when including $\{e\}$. By this pattern, it can also be concluded that maximal cliques of $\Gamma_{D_{2n}}^{cp}$ that do not involve $\{e\}$ do not exist.

Hence, a recursive relationship can be established between the cliques that include $\{e\}$ and those that do not. Let c_k denote the number of size-k cliques, c_k^e be the number of size-k cliques with $\{e\}$ and c_k' be the number of size-k cliques without $\{e\}$. Then

$$c_k = c_k^e + c_k', \text{ with } c_k' \equiv c_{k+1}^e, \text{ and } c_0^e = c_\omega' = 0, \text{ for } 0 \leq k \leq \omega,$$

where ω is clique number of the coprime graph of dihedral groups. Since $\omega = \omega(\Gamma_{D_{2n}}^{cp}) = m + 2$, where m is a non-negative integer, the alternating sum over all clique sizes of $\Gamma_{D_{2n}}^{cp}$ is calculated as follows

$$\begin{aligned}\sum_{k=0}^{\omega}(-1)^k c_k &= c_0 - c_1 + c_2 - c_3 + \cdots + c_m - c_{m+1} + c_{m+2} \\ &= (c_0^e + c_0') - (c_1^e + c_1') + (c_2^e + c_2') - (c_3^e + c_3') + \cdots + (c_m^e + c_m') - (c_{m+1}^e + c_{m+1}') \\ &\quad + (c_{m+2}^e + c_{m+2}') \\ &= (c_0^e + c_1^e) - (c_1^e + c_2^e) + (c_2^e + c_3^e) - (c_3^e + c_4^e) + \cdots + (c_m^e + c_{m+1}^e) \\ &\quad - (c_{m+1}^e + c_{m+2}^e) + (c_{m+2}^e + c_{m+2}') \\ &= c_0^e + c_{m+2}' \\ &= 0.\end{aligned}$$

Suppose the result for $\omega = m + 2$ is true, then for $\omega = m + 3$,

$$\begin{aligned}\sum_{k=0}^{m+3}(-1)^k c_k &= \left(\sum_{k=0}^{m+2}(-1)^k c_k\right) - c_{m+3} \\ &= (c_0^e + c_{m+2}') - (c_{m+3}^e + c_{m+3}') \\ &= (c_0^e + c_{m+3}^e) - (c_{m+3}^e + c_{m+3}') \\ &= c_0^e - c_{m+3}' \\ &= 0.\end{aligned}$$

Thus, by induction method, the alternating sum is always equal to zero for any clique number of the coprime graph of dihedral groups $\omega(\Gamma_{D_{2n}}^{cp})$. Analyzing further when $\omega(\Gamma_{D_{2n}}^{cp})$ is an odd number or an even number, the result is generalized as

$$\sum_{j=0}^{\lfloor\frac{\omega}{2}\rfloor} c_{2j} = \sum_{j=0}^{\lfloor\frac{\omega-1}{2}\rfloor} c_{2j+1}\,,$$

where $\lfloor\cdot\rfloor$ is the floor function and c_k is the number of size-k clique in $\Gamma_{D_{2n}}^{cp}$ and k is a non-negative integer.

The next four propositions are shown to support the odd-even clique equality property mentioned in Theorem 3 as the clique polynomial of the coprime graph of dihedral groups will be determined for some cases, which are when $n = 2^{k_0}$, $n = p^{k_0}$, $n = 2p$ and $n = pq$.

Proposition 4
Let $\Gamma_{D_{2n}}^{cp}$ be the coprime graph of dihedral groups, for $n = 2^{k_0}$, where $k_0 \in N$. Then, the clique polynomial of $\Gamma_{D_{2n}}^{cp}$ satisfies the odd-even clique equality.

Proof. When $n = 2^{k_0}$, the elements in D_{2n} can be partitioned into two independent sets, $V_1 = \{e\}$ and $V_2 = D_{2n}\backslash\{e\}$. The all elements in V_2 have orders of 2^l for some $l \in N$, implies these elements cannot form any edge in the coprime graph. Furthermore, as stated in Theorem 3, $\{e\}$ is connected to all the other vertices in $\Gamma_{D_{2n}}^{cp}$. Therefore, $\Gamma_{D_{2n}}^{cp}$ must be a complete bipartite graph and its clique polynomial is $C(\Gamma_{D_{2n}}^{cp}; x) = 1 + 2nx + (2n-1)x^2$. By comparing the coefficients, $c_0 + c_2 = c_1$.

Proposition 5
Let $\Gamma^{cp}_{D_{2n}}$ be the coprime graph of dihedral groups, for $n = p^{k_0}$, where p is an odd prime and $k_0 \in N$. Then, the clique polynomial of $\Gamma^{cp}_{D_{2n}}$ satisfies the odd-even clique equality.

Proof. When $n = p^{k_0}$, the elements in D_{2n} can be partitioned into three independent sets, $V_1 = \{e\}$, V_2 be the set of elements of order 2 and V_3 be the set of elements of order p^l for some $l \in N$. Similarly, $\{e\}$ connected to all the other vertices implies there are also $2n - 1$ number of size-2 cliques.

However, there now exist size-2 cliques formed by the elements $u \in V_2$ and $v \in V_3$ since all elements in V_2 are coprime to elements in V_3. As both V_2 and V_3 are vertex disjoint sets, the cliques of $\{u, v\}, \forall u \in V_2$ and $\forall v \in V_3$ can be represented by the joint of graphs $V_2 \vee V_3$. Let $n_1 = |(V_2 \vee V_3)|$, then based on Theorem 2, there exists also n_1 number of size-3 cliques by involving $V_1 = \{e\}$. The size-3 cliques are denoted by $\{e, u, v\} \in V_1 \vee V_2 \vee V_3$, where $e \in V_1, \forall u \in V_2$ and $\forall v \in V_3$, which are also all possible maximal cliques in $\Gamma^{cp}_{D_{2n}}$.

Therefore, $\Gamma^{cp}_{D_{2n}}$ must be a complete tripartite graph and its clique polynomial is $C(\Gamma^{cp}_{D_{2n}}; x) = 1 + 2nx + (2n - 1 + n_1)x^2 + n_1x^3$. By comparing the coefficients, $c_0 + c_2 = c_1 + c_3$.

Proposition 6
Let $\Gamma^{cp}_{D_{2n}}$ be the coprime graph of dihedral groups, for $n = 2p$, where p is an odd prime. Then, the clique polynomial of $\Gamma^{cp}_{D_{2n}}$ satisfies the odd-even clique equality.

Proof. When $n = 2p$, the elements in D_{2n} are partitioned into four independent sets instead of three, such that $V_1 = \{e\}$, V_2 be the set of elements of order 2 and 4, V_3 be the set of elements of order p and V_4 be the set of elements of order $2p$. Based on Theorem 2, the clique number should be equal to 3. Based on Proposition 5, the cliques are listed accordingly as shown in Table 1 below.

Table 1: Number of cliques of $\Gamma^{cp}_{D_{2n}}$ up to size-3, when $n = 2p$

Clique size	Cliques
0	$\{\}$
1	$V_1 \cup V_2 \cup V_3 \cup V_4 = V(\Gamma^{cp}_{D_{2n}})$
2	$(V_1 \vee V_2) \cup (V_1 \vee V_3) \cup (V_1 \vee V_4) \cup (V_2 \vee V_3)$
3	$(V_1 \vee V_2 \vee V_3)$

From table 1, it is clear that size-0 cliques $c_0 = 1$ and size-1 cliques $c_1 = 2n$. Meanwhile, $|(V_1 \vee V_2) \cup (V_1 \vee V_3) \cup (V_1 \vee V_4)| = |V(\Gamma^{cp}_{D_{2n}}) \backslash \{e\}| = 2n - 1$. Let $|(V_2 \vee V_3)| = n_2$, then $c_2 = 2n - 1 + n_2$. Based on Theorem 3, $|(V_1 \vee V_2 \vee V_3)| = |(V_2, V_3)|$, then $c_3 = n_2$. Therefore, the clique polynomial is $C(\Gamma^{cp}_{D_{2n}}; x) = 1 + 2nx + (2n - 1 + n_2)x^2 + n_2x^3$. By comparing the coefficients, $c_0 + c_2 = c_1 + c_3$.

Proposition 7
Let $\Gamma^{cp}_{D_{2n}}$ be the coprime graph of dihedral groups, for $n = pq$, where p and q are distinct odd primes. Then, the clique polynomial of $\Gamma^{cp}_{D_{2n}}$ satisfies the odd-even clique equality.

Proof. When $n = pq$, the elements in D_{2n} are partitioned into four independent sets instead of three, such that $V_1 = \{e\}$, V_2 be the set of elements of order 2, V_3 be the set of elements of order p, V_4 be the set of elements of order q and V_5 be the set of elements of order pq. Similar to Proposition 6, the cliques are then listed accordingly and shown in Table 2 below.

Table 2: Number of cliques of $\Gamma_{D_{2n}}^{cp}$, when $n = pq$

Clique size	Cliques
0	$\{\}$
1	$V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 = V(\Gamma_{D_{2n}}^{cp})$
2	$(V_1 \vee V_2) \cup (V_1 \vee V_3) \cup (V_1 \vee V_4) \cup (V_1 \vee V_5) \cup (V_2 \vee V_3) \cup (V_2 \vee V_4) \cup (V_2 \vee V_5) \cup (V_3 \vee V_4)$
3	$(V_1 \vee V_2 \vee V_3) \cup (V_1 \vee V_2 \vee V_4) \cup (V_1 \vee V_2 \vee V_5) \cup (V_1 \vee V_2 \vee V_4) \cup (V_2 \vee V_3 \vee V_4)$
4	$(V_1 \vee V_2 \vee V_3 \vee V_4)$

The method of calculation is similar as shown in Proposition 5. Then, $|(V_1 \vee V_2) \cup (\{V_1 \vee V_3) \cup (V_1 \vee V_4) \cup (V_1 \vee V_5)| = 2n - 1$ and let $|(V_2 \vee V_3) \cup (V_2 \vee V_4) \cup (V_2 \vee V_5) \cup (V_3 \vee V_4)| = n_3$ and $|(V_2 \vee V_3 \vee V_4)| = n_4$. Hence, the clique polynomial obtained is $C(\Gamma_{D_{2n}}^{cp}; x) = 1 + 2nx + (2n - 1 + n_3)x^2 + (n_3 + n_4)x^3 + n_4x^4$. By comparing the coefficients, $c_0 + c_2 + c_4 = c_1 + c_3$.

Conclusion

The odd-even clique equality property of the coprime graph of the dihedral groups is established, regardless of the group order for D_{2n}. The clique polynomials are also determined for the coprime group of specific cases of dihedral groups which include, when $n = 2^{k_0}, n = p^{k_0}, n = 2p$ and $n = pq$. Overall, this clique property highlights the symmetry of clique distribution of the coprime graph associated to dihedral groups.

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