



Proceedings of Science and Mathematics

Faculty of Science,
Universiti Teknologi Malaysia

Volume 34, 2026, page 46-50

The Idempotent-Unit Action Graphs Associated with Some Rings with Unity

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Abstract

This paper investigates the idempotent-unit action graph associated with some finite rings with unity. Let R be a finite ring, $Id(R)$ be the set of idempotent elements of R and $U(R)$ be the set of unit elements of R . The idempotent-unit action graph, $\Gamma_{IU}(R)$, is defined as a simple undirected graph whose vertices are all elements of R , and two distinct vertices x and y are adjacent whenever $x - y = uz$ or $x - y = zu$ for some $z \in Id(R)$ and $u \in U(R)$. The idempotent-unit action graphs are constructed for rings \mathbb{Z}_p and \mathbb{Z}_p^n for $n \in \mathbb{Z}$ and $n \geq 2$. In particular, we show that $\Gamma_{IU}(\mathbb{Z}_p)$, and $\Gamma_{IU}(\mathbb{Z}_{pq})$ are complete graphs, $\Gamma_{IU}(\mathbb{Z}_p^n)$ is a complete p -partite graph with equal part sizes p^{n-1} .

Keywords: Idempotent-unit action graph, commutative rings, graph theory, partite graph, idempotents, units.

Introduction

Graphs associated with algebraic structures have become an important tool for translating algebraic information into combinatorial language (Bera, 2018). The power graphs, commuting graphs, and enhanced power graphs of finite groups and rings, each encoding different interaction patterns among elements (Moghaddamfar et al., 2014). These graphs allow one to study algebraic properties through graph-theoretic invariants such as connectivity, diameter, and chromatic number, and conversely to construct graphs with prescribed combinatorial features from algebraic data (Aalipour & Akbari, 2013).

For finite groups, enhanced power graphs and their variants have been widely studied, with detailed structural descriptions and applications to several classes of groups, including metabelian and dihedral groups (Kumar et al., 2023). Similar ideas have been extended to rings, leading to a variety of graphs defined using unit elements, zero-divisors, idempotents, and ideals (Djuang et al., 2025). In particular, idempotent elements play a central role in decomposing finite commutative rings (Debnath et al., 2024), while unit elements capture multiplicative symmetries. Thus, it is natural to consider graphs built from their interaction (Djuang et al., 2025).

The unitary Cayley graph of a ring R , is a graph where vertices are elements of the ring and an edge exists between x and y if $x - y \in U(R)$, where $U(R)$ is the set of unit elements in R (Dejter, 1995). This structure has been extensively studied for the rings of integers modulo n . In this research, a new defined graph related to rings is given, namely the idempotent-unit action graph, $\Gamma_{IU}(R)$, for some finite rings R with unity. The vertices of $\Gamma_{IU}(R)$ are all elements of R , and two distinct vertices are adjacent whenever their difference can be expressed as a product of an idempotent element with a unit element.

The aim of this paper is to focus on four classes of rings, which are $\mathbb{Z}_p, \mathbb{Z}_{p^n}, \mathbb{Z}_{pq}$ and \mathbb{Z}_{p^nq} , where $n \in \mathbb{Z}^+, n \geq 2$ and $p < q, p$ and q are primes.

Preliminaries

In this section, some basic concepts of ring theory and graph theory are given starting with two definitions that describes the idempotent elements and units of a ring, respectively followed by a basic theorem on idempotents.

Definition 1 [Gallian, 2021] An element z in a ring R is called idempotent if $z^2 = z$. The set of all idempotents of R is denoted by $Id(R)$.

Definition 2 [Gallian, 2021] An element u in a ring R with unity is called a unit if there exists $v \in R$ such that $uv = 1$. The set of all units of R denoted by $U(R)$.

Theorem 1 [Levaku, 2025] Let $n \geq 1$ be an integer. Then, the number of idempotent elements in the ring $(\mathbb{Z}_n, \oplus, \odot)$ is 2^k , where k is the number of distinct prime divisors of n .

Next, some preliminaries on graphs are given in the following definitions. Note that all graphs considered in this paper are finite, simple and undirected. Notes that, $V(\Gamma)$ and $E(\Gamma)$ denotes a vertex set and an edge set of a graph Γ , respectively.

Definition 3 [Balakrishnan, 2012] Two vertices, v_1 and v_2 in $V(\Gamma)$ are said to be adjacent, $v_1 \sim v_2$ if there exists an edge $e_1 \in E(\Gamma)$ so that $e_1 = \{v_1, v_2\}$.

Definition 4 [Wilson, 2015] A graph Γ is known as a complete graph, K_n , provided that every vertex $v \in V(\Gamma)$ is connected to other vertices in $V(\Gamma)$.

Definition 5 [West, 2001] A k -partite graph is a graph whose vertex set can be partitioned into k independent sets.

A complete k -partite graph is a k -partite graph in which two vertices are adjacent if and only if they belong to different partite sets.

Definition 6 [Bondy, 1976] If all vertices in Γ have the same degree, that is, $\deg(v) = r$ for $v \in V(\Gamma)$, then, the graph Γ is called r -regular; otherwise, it is called irregular.

Results and Discussion

This section provides a formal new definition of the idempotent-unit action graph of a ring with unity, $\Gamma_{IU}(R)$, that are used to construct the graph for $\mathbb{Z}_p, \mathbb{Z}_{p^n}, \mathbb{Z}_{pq}$ and \mathbb{Z}_{p^nq} . Then, several properties are shown for $\Gamma_{IU}(\mathbb{Z}_p), \Gamma_{IU}(\mathbb{Z}_{pq})$ and $\Gamma_{IU}(\mathbb{Z}_{p^n})$. The construction of the idempotent-unit action graph for \mathbb{Z}_5 is shown in Example 1, along with the general presentation for $\Gamma_{IU}(\mathbb{Z}_p)$ in Proposition 1.

Definition 12 Idempotent-Unit Action Graph

Let R be a ring with unity, $Id(R)$ be the set of idempotent elements of R and $U(R)$ be the set of unit elements of R . The idempotent-unit action graph of a ring R , denoted by $\Gamma_{IU}(R)$, is a graph with R as the vertex set and two distinct vertices x and y are adjacent if there exist $z \in Id(R) \setminus \{0\}$ and $u \in U(R)$ such that

$$x - y = uz \text{ or } x - y = zu.$$

Remarks: In a commutative ring, only the first condition is used since $uz = zu$.

Example 1 Consider the ring of integers modulo 5, \mathbb{Z}_5 . By Definition 12, $V(\Gamma_{IU}(\mathbb{Z}_5)) = \{0,1,2,3,4\}$. The set of idempotent elements of \mathbb{Z}_5 is $Id(\mathbb{Z}_5) = \{0,1\}$, since $0^2 = 0$ and $1^2 = 1(mod 5)$. The set of unit elements of \mathbb{Z}_5 is $U(\mathbb{Z}_5) = \{1,2,3,4\}$, consisting of all nonzero elements in \mathbb{Z}_5 . Two distinct vertices x and y are adjacent if there exist $z \in Id(\mathbb{Z}_5) \setminus \{0\}$ and $u \in U(\mathbb{Z}_5)$ such that $x - y = uz$. Here, the possible values of $x - y(mod 5)$ and $uz(mod 5)$ are 0, 1, 2, 3 or 4. Thus, all vertices of $\Gamma_{IU}(\mathbb{Z}_5)$ are adjacent. Hence, $\Gamma_{IU}(\mathbb{Z}_5) = K_5$.

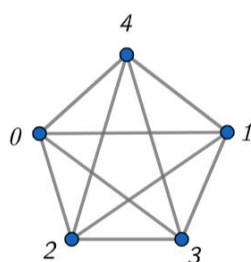


Figure 1 The idempotent-unit action graph of \mathbb{Z}_5

Following this example, the idempotent-unit action graph for any prime p is presented in Proposition 1.

Proposition 1 If p is a prime number. Then, $\Gamma_{IU}(\mathbb{Z}_p) = K_p$.

Proof. Let p be any prime number. Then, \mathbb{Z}_p is a field, hence, every non-zero elements is a unit. Thus, $U(\mathbb{Z}_p) = \mathbb{Z}_p \setminus \{0\}$. An element in \mathbb{Z}_p is idempotent if and only if $z^2 = z$. This is equivalent to $z(z - 1) = 0$. Since \mathbb{Z}_p has no zero divisor, this implies $z = 0$ or $z = 1$, but then we exclude 0 since $z \in Id(R) \setminus \{0\}$. Therefore, $Id(R) \setminus \{0\} = \{1\}$.

Now, let $x, y \in \mathbb{Z}_p$ where $x \neq y$. Then, $x - y \neq 0$, so $x - y \in \mathbb{Z}_p \setminus \{0\} = U(\mathbb{Z}_p)$. Now, by taking $u = x - y \in U(\mathbb{Z}_p)$ and $z = 1 \in Id(R) \setminus \{0\}$, we have $x - y = uz$. Hence, x and y are adjacent in $\Gamma_{IU}(\mathbb{Z}_p)$. The vertex set is \mathbb{Z}_p , which have p elements. Thus, $\Gamma_{IU}(\mathbb{Z}_p)$ is the complete graph on p vertices, that is $\Gamma_{IU}(\mathbb{Z}_p) = K_p$. □

The idempotent-unit action graph for \mathbb{Z}_9 is constructed in the next example along with the general presentation for $\Gamma_{IU}(\mathbb{Z}_{p^n})$, for $n \geq 2$ in Proposition 2.

Example 2. Consider a ring \mathbb{Z}_9 . By Definition 12, $V(\Gamma_{IU}(\mathbb{Z}_9)) = \{0,1,2,3,4,5,6,7,8\}$, which consists of all elements in \mathbb{Z}_9 . The set of idempotent elements of \mathbb{Z}_9 is $Id(\mathbb{Z}_9) = \{0,1\}$, since $0^2 = 0$ and $1^2 = 1(mod 9)$. The set of unit elements of \mathbb{Z}_9 is $U(\mathbb{Z}_9) = \{1,2,4,5,7,8\}$. Furthermore, $\{uz|u \in U(\mathbb{Z}_9), z \in Id(\mathbb{Z}_9) \setminus \{0\}\} = \{1,2,4,5,7,8\}$. Two distinct vertices x and y are connected if there exist $z \in Id(\mathbb{Z}_9) \setminus \{0\}$ and $u \in U(\mathbb{Z}_9)$ such that $x - y = uz$. Hence, the adjacencies are given as follows:

$$\begin{aligned} 0 &\sim \{1,2,4,5,7,8\}, & 1 &\sim \{0,2,3,5,6,8\}, \\ 2 &\sim \{0,1,3,4,6,7\}, & 3 &\sim \{1,2,4,5,7,8\}, \end{aligned}$$

$$\begin{aligned} 4 &\sim \{0,2,3,5,6,8\}, & 5 &\sim \{0,1,3,4,6,7\}, \\ 6 &\sim \{1,2,4,5,7,8\}, & 7 &\sim \{0,2,3,5,6,8\}, \\ 8 &\sim \{0,1,3,4,6,7\}. \end{aligned}$$

Therefore, $\Gamma_{IU}(\mathbb{Z}_9) = K_{3,3,3}$.

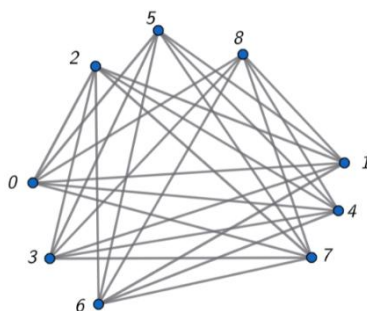


Figure 2 The idempotent-unit action graph of \mathbb{Z}_9

Proposition 2 If p is a prime number and $n \in \mathbb{Z}^+$, where $n \geq 2$. Then, $\Gamma_{IU}(\mathbb{Z}_{p^n}) = K_{\underbrace{p^{n-1}, p^{n-1}, \dots, p^{n-1}}_{p \text{ times}}}$.

Proof Let p be any prime number and $n \in \mathbb{Z}^+$ such that $n \geq 2$. Thus, \mathbb{Z}_{p^n} is a ring with unique maximal ideal $\langle p \rangle$ and an element $a \in \mathbb{Z}_{p^n}$ is a unit if and only if $\gcd\{a, p\} = 1$. Hence, $U(\mathbb{Z}_{p^n}) = \{a \in \mathbb{Z}_{p^n} \mid p \nmid a\}$. Next, an element $b \in \mathbb{Z}_{p^n}$ is idempotent if and only if $b^2 \equiv b \pmod{p^n}$, i.e. $b(b - 1) \equiv 0 \pmod{p^n}$. Since the only prime divisor of p^n is p , this implies $b \equiv 0$ or $b \equiv 1 \pmod{p^n}$. Thus, $Id(\mathbb{Z}_{p^n}) = \{0, 1\}$.

Furthermore, $\{uz \mid u \in U(\mathbb{Z}_{p^n}), z \in Id(\mathbb{Z}_{p^n}) \setminus \{0\}\} = \{u \cdot 1 \mid u \in U(\mathbb{Z}_{p^n})\} = U(\mathbb{Z}_{p^n})$. By Definition 12, two distinct vertices $x, y \in \mathbb{Z}_{p^n}$ are adjacent if and only if $x - y \in U(\mathbb{Z}_{p^n})$, that is if and only if

$$x \sim y \Leftrightarrow p \nmid (x - y). \tag{3.1}$$

In particular, adjacency depends only on whether x and y are congruent modulo p . Now, partition \mathbb{Z}_{p^n} into its residue classes modulo p , $p \nmid (x - y)$.

$$C_r = \{x \in \mathbb{Z}_{p^n} \mid x \equiv r \pmod{p}\}, r = 0, 1, 2, \dots, p - 1.$$

These sets are pairwise disjoint and $\mathbb{Z}_{p^n} = \bigcup_{r=0}^{p-1} C_r$. Each class C_r contains exactly p^{n-1} elements, since $|\mathbb{Z}_{p^n}| = p^n$ and there are p residue classes modulo p . Now, these C_r form the partite sets of a complete p -partite graph. Let $x, y \in C_r$ with $x \neq y$. Then, $x \equiv y \pmod{p}$, so $x - y \equiv 0 \pmod{p}$. Hence, $p \mid (x - y)$. By (3.1), $x \not\sim y$. Thus, no two distinct vertices in the same C_r are adjacent, so each C_r is an independent set.

Next, let $x \in C_r$ and $y \in C_s$ with $r \neq s$. Then, $x \neq y \pmod{p}$, so $x - y \neq 0 \pmod{p}$. Then, by (3.1) $p \nmid (x - y)$. This implies $x \sim y$. Hence, every vertex in C_r is adjacent to every vertex in C_s . Thus, the vertex set of $\Gamma_{IU}(\mathbb{Z}_p^n)$ is partitioned into the p independent sets C_0, C_1, \dots, C_{p-1} , each of size $|C_r| = p^{n-1}$, and every edge joins vertices from different sets. This is the complete p -partite graph with part sizes p^{n-1} . Therefore, $\Gamma_{IU}(\mathbb{Z}_p^n) = K \underbrace{p^{n-1}, p^{n-1}, \dots, p^{n-1}}_{p \text{ times}}$. \square

Conclusion

In this paper, the idempotent-unit action graph, $\Gamma_{IU}(R)$, is constructed for some commutative rings with unity, which are \mathbb{Z}_p and \mathbb{Z}_p^n . It is proven that $\Gamma_{IU}(\mathbb{Z}_p)$ is a complete graph, while \mathbb{Z}_p^n with $n \geq 2$ is a complete p -partite graph with equal part sizes p^{n-1} . In the case of \mathbb{Z}_p and \mathbb{Z}_p^n , the graphs obtained are isomorphic to the unitary Cayley graph.

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