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### On the Harary Index of the Non-Normal Cyclic Subgroup Graph Associated to the Dihedral Groups of Certain Orders

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#### Abstract

Topological indices are widely used in chemistry as molecular descriptors for analyzing molecular structures and predicting their chemical and physical properties. Among these indices, the Harary index has emerged as one of the most extensively studied measures in mathematical chemistry and related disciplines, as it provides a convenient representation of molecular compactness. While extensive research has been conducted on the Harary index of undirected graphs, investigations involving directed graphs remain limited. In this study, the Harary index is explored in the setting of directed graphs (digraphs) arising from algebraic structures. Specifically, we consider the digraph of the non-normal cyclic subgroups associated to the dihedral groups of order six and eight. By examining the reciprocal distances in these digraphs for the dihedral groups, the corresponding Harary indices are determined explicitly. The results show that the Harary index increases as the order of the digraph increases. These findings offer new insights into the behavior of the Harary index in directed graph settings and highlight its potential for further applications in algebraic graph theory.

**Keywords:** Digraph; non-normal cyclic subgroup; Harary index

#### Introduction

A topological index in mathematical chemistry is defined as a numerical invariant derived from the structural diagram of a chemical compound and is commonly used to characterize properties of molecular graphs [1, 2]. Among these indices, the Harary index is one of the most extensively studied in mathematical chemistry and related disciplines. It was introduced independently by Plavšić *et al.* [3] in 1993 and is known to possess several interesting properties, particularly as a measure of molecular compactness. The Harary index and its variants have been successfully applied to model eight physical properties, including boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures, surface tensions, and melting points of 74 alkanes from ethane to nonane [3]. The extension of the Harary index to digraphs was first studied by Jiang *et al.* [4] in 2015, wherein only finite and simple digraphs were considered.

In this study, the Harary index is determined for digraphs of non-normal cyclic subgroups associated to the dihedral groups of orders six and eight. Since this study examines the digraphs associated to dihedral groups; thus, the symmetry set of a regular  $n$ -gon, denoted by  $D_{2n}$ , represents a dihedral group of order  $2n$  for any integers  $n$ . The study of digraphs of non-normal cyclic subgroups was first introduced by Rahin *et al.* [5] in 2020 in the context of dihedral groups and was later extended

to generalized quaternion groups [6]. The Harary index is obtained by examining the distances and reciprocal distances between all ordered pairs of vertices within each component of the digraphs, from which the Harary indices are determined.

This paper is organized as follows. Section 1 provides a brief review of earlier studies on the Harary index of graphs and digraphs. Section 2 introduces the preliminary concepts required for this study. The main results are presented in Section 3 in the form of propositions. Finally, Section 4 summarizes the findings and provides concluding remarks.

**Preliminaries**

Some relevant definitions and properties used in this study are presented in this section.

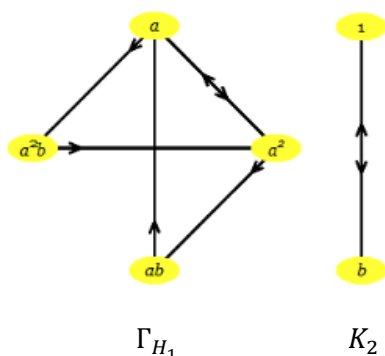
**Definition 1** [5] Non-Normal Cyclic Subgroup Graph

Let  $H$  be a non-normal subgroup of a group  $G$ . The subgroup graph  $\Gamma_H^{NN}(G)$  is defined as a digraph whose vertex set consists of all elements of  $G$ . A directed edge from a vertex  $x$  to a vertex  $y$  exists if and only if  $x \neq y$  and  $xy \in H$ .

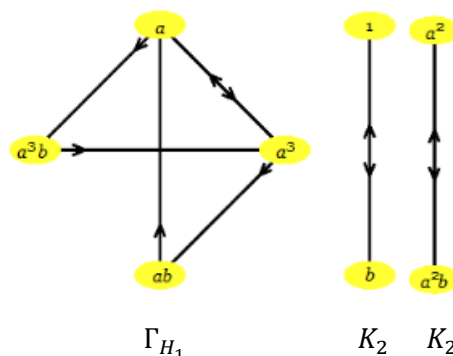
Examples of these digraphs are given next.

**Example 1** [5]

Consider  $D_6$  and  $D_8$  that correspond to the dihedral groups,  $D_{2n}$  with  $n = 3$  and  $n = 4$ , respectively, and  $H_1 = \langle b \rangle = \{1, b\}$  is a non-normal subgroup. The subgroup graphs  $\Gamma_{H_1}^{NN}(D_6)$  and  $\Gamma_{H_1}^{NN}(D_8)$  are digraphs illustrated in Figure 1 and Figure 2.



**Figure 1**  $\Gamma_{H_1}^{NN}(D_6)$



**Figure 2**  $\Gamma_{H_1}^{NN}(D_8)$

**Definition 2** [7] Distance Between Two Vertices

The shortest path from vertex  $u$  to vertex  $v$  in a graph is called the distance between the two vertices  $u$  and  $v$ , denoted as  $d(u, v)$ . If no path exists, then the distance is zero.

**Definition 3** [4] Reciprocal Distance Between Two Vertices

The reciprocal distance between two vertices of a digraph  $\Gamma$ , denoted by  $Rd(u, v)$ , is defined as follows:

$$Rd(u, v) = \begin{cases} \frac{1}{d(u, v)}, & \text{if } u \neq v, \\ 0, & \text{if } u = v. \end{cases}$$

**Definition 4** [4] Harary Index of Digraphs

Let  $\Gamma$  be a digraph and  $Rd(u, v)$  represent the reciprocal distance between the vertices  $u$  and  $v$  in  $\Gamma$ . Then, the Harary index of  $\Gamma$ , denoted by  $H(\Gamma)$ , is written as

$$H(\Gamma) = \frac{1}{2} \sum_{(u,v) \in V(\Gamma) \times V(\Gamma)} Rd(u, v).$$

**Definition 5** [8] Harary Index of Disconnected Graph

Let  $R$  be a disconnected graph with  $R_1, R_2, \dots, R_k$  as its components. The Harary index of  $R$ ,  $H(R)$ , is written as

$$H(R) = \sum_{i=1}^k H(R_i).$$

**Main Results**

In this section, the Harary index of the digraph of the non-normal cyclic subgroup  $H_1$  of dihedral groups of orders six,  $\Gamma_{H_1}^{NN}(D_6)$ , and eight,  $\Gamma_{H_1}^{NN}(D_8)$ , are determined. First, the distances and reciprocal distances are computed and then used in the proofs to derive the corresponding Harary indices, stated as propositions.

**Proposition 1** Let  $D_6$  be the dihedral group of order six and  $\Gamma_{H_1}^{NN}(D_6)$  be the digraph of the non-normal cyclic subgroup associated to  $D_6$ . Then, the Harary index of  $\Gamma_{H_1}^{NN}(D_6)$  is  $H(\Gamma_{H_1}^{NN}(D_6)) = \frac{11}{2}$ .

**Proof.** Let  $\Gamma_{H_1}^{NN}(D_6)$  denote the digraph of the non-normal cyclic subgroup associated to the dihedral group of order six,  $D_6$ , as shown in Figure 1. From Figure 1, it is observed that  $\Gamma_{H_1}^{NN}(D_6) = \Gamma_{H_1} \cup K_2$ , hence consists of two components, namely  $\Gamma_{H_1}$  and  $K_2$ . By Definition 4, the Harary index of a digraph  $\Gamma_{H_1}^{NN}(D_6)$  is defined as half of the sum of the reciprocal distances over all ordered pairs of vertices. Furthermore, according to Definition 5, the Harary index of a disconnected graph is equal to the sum of the Harary indices of its components. Consequently, the distances and reciprocal distances corresponding to each distinct ordered pair of vertices within each component of  $\Gamma_{H_1}^{NN}(D_6)$  are computed using Definition 2 and Definition 3. The resulting distances and reciprocal distances are presented in Table 1.

**Table 1:** The distances and reciprocal distances of  $(u, v)$  in  $\Gamma_{H_1}^{NN}(D_6)$ .

Component	$(u, v)$	$d(u, v)$	$Rd(u, v)$	$(u, v)$	$d(u, v)$	$Rd(u, v)$
$K_2$	$(1, b)$	1	1	$(b, 1)$	1	1
	$(a, a^2)$	1	1	$(ab, a)$	1	1
	$(a, ab)$	2	$\frac{1}{2}$	$(ab, a^2)$	2	$\frac{1}{2}$
$\Gamma_{H_1}$	$(a, a^2b)$	1	1	$(ab, a^2b)$	2	$\frac{1}{2}$
	$(a^2, a)$	1	1	$(a^2b, a)$	2	$\frac{1}{2}$
	$(a^2, ab)$	1	1	$(a^2b, a^2)$	1	1

$(a^2, a^2b)$	2	$\frac{1}{2}$	$(a^2b, ab)$	2	$\frac{1}{2}$
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Subsequently, based on all the computed reciprocal distances, the Harary index for the component  $K_2$  is obtained as

$$H(K_2) = \frac{1}{2} \sum_{(u,v) \in V(K_2) \times V(K_2)} Rd(u, v) = \frac{1}{2}(1 + 1) = 1.$$

Meanwhile, the Harary index for the component  $\Gamma_{H_1}$  is obtained as

$$\begin{aligned} H(\Gamma_{H_1}) &= \frac{1}{2} \sum_{(u,v) \in V(\Gamma_{H_1}) \times V(\Gamma_{H_1})} Rd(u, v) \\ &= \frac{1}{2} \left( 1 + \frac{1}{2} + 1 + 1 + 1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} \right) \\ &= \frac{9}{2}. \end{aligned}$$

Therefore, by Definition 5, the Harary index of  $\Gamma_{H_1}^{NN}(D_6)$  is obtained by summing the Harary indices of  $\Gamma_{H_1}$  and  $K_2$ . Thus,

$$H(\Gamma_{H_1}^{NN}(D_6)) = H(\Gamma_{H_1}) + H(K_2) = \frac{9}{2} + 1 = \frac{11}{2}.$$

Next, the Harary index of  $\Gamma_{H_1}^{NN}(D_8)$  is established in the following proposition.

**Proposition 2** Let  $D_8$  be the dihedral group of order eight and  $\Gamma_{H_1}^{NN}(D_8)$  be the digraph of the non-normal cyclic subgroup associated to  $D_8$ . Then, the Harary index of  $\Gamma_{H_1}^{NN}(D_8)$  is  $H(\Gamma_{H_1}^{NN}(D_8)) = \frac{13}{2}$ .

**Proof.** Let  $\Gamma_{H_1}^{NN}(D_8)$  be the digraph of the non-normal cyclic subgroup associated with  $D_8$ , as shown in Figure 2. It follows from Figure 2 that  $\Gamma_{H_1}^{NN}(D_8) = \Gamma_{H_1} \cup K_2 \cup K_2$ . Next, the distances and reciprocal distances for all ordered distinct pairs of vertices within each component are computed using Definition 2 and Definition 3, and the results are summarized in Table 2.

**Table 2:** The distances and reciprocal distances of  $(u, v)$  in  $\Gamma_{H_1}^{NN}(D_8)$ .

Component	$(u, v)$	$d(u, v)$	$Rd(u, v)$	$(u, v)$	$d(u, v)$	$Rd(u, v)$
$K_2$	$(1, b)$	1	1	$(b, 1)$	1	1
$K_2$	$(a^2, a^2b)$	1	1	$(a^2b, a^2)$	1	1
	$(a, a^3)$	1	1	$(ab, a)$	1	1
	$(a, ab)$	2	$\frac{1}{2}$	$(ab, a^3)$	2	$\frac{1}{2}$
$\Gamma_{H_1}$	$(a, a^3b)$	1	1	$(ab, a^3b)$	2	$\frac{1}{2}$
	$(a^3, a)$	1	1	$(a^3b, a)$	2	$\frac{1}{2}$
	$(a^3, ab)$	1	1	$(a^3b, a^3)$	1	1

$$\begin{array}{ccccccc} (a^3, a^3b) & 2 & \frac{1}{2} & (a^3b, ab) & 2 & \frac{1}{2} \\ \hline \end{array}$$

From the computed reciprocal distances, it follows that the Harary indices of the two components of  $K_2$  are obtained as

$$\begin{aligned} 2(H(K_2)) &= 2 \left( \frac{1}{2} \sum_{(u,v) \in V(K_2) \times V(K_2)} Rd(u, v) \right) \\ &= 2 \left( \frac{1}{2} (1 + 1) \right) \\ &= 2. \end{aligned}$$

Next, the Harary index for the component  $\Gamma_{H_1}$  is obtained as

$$\begin{aligned} H(\Gamma_{H_1}) &= \frac{1}{2} \sum_{(u,v) \in V(\Gamma_{H_1}) \times V(\Gamma_{H_1})} Rd(u, v) \\ &= \frac{1}{2} \left( 1 + \frac{1}{2} + 1 + 1 + 1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} \right) \\ &= \frac{9}{2}. \end{aligned}$$

Hence, by Definition 5, the Harary index of  $\Gamma_{H_1}^{NN}(D_8)$  is

$$H(\Gamma_{H_1}^{NN}(D_8)) = H(\Gamma_{H_1}) + 2(H(K_2)) = \frac{9}{2} + 2 = \frac{13}{2}.$$

By applying the same method described in Proposition 1 and Proposition 2, the Harary index can be determined for any digraph of the non-normal cyclic subgroup associated to dihedral groups of higher order.

### Conclusion

In this research, the Harary index has been determined for the digraphs of non-normal cyclic subgroups associated to the dihedral groups of orders six and eight. The results indicate that the Harary index increases as the order of the group increases, with rational values obtained in both cases. As a suggestion for future work, the approach presented in this study is expected to be applicable to the determination of the general form of the Harary index for this class of digraphs.

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