Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Volume 34, 2026, page 57-61

Matrices of Structure Constants for the Identities of Finite-Dimensional Commutative and Associative Algebras

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Abstract

The classification of finite-dimensional algebras is a fundamental problem in modern algebra, where polynomial identities playing a crucial role in distinguishing algebraic structures. Using the matrix of structure constants (MSC) approach, an n-dimensional algebra with a fixed basis is represented by a matrix A, whose entries are the structure constants of the linear combinations of its basis vectors. Although many classes of algebra are defined by polynomial identities in abstract vectors, these identities do not directly yield the equation systems required for classification. In this paper, we bridge this gap by introducing a coordinate-based identity conversion procedure within the MSC framework for commutative and associative algebras. Polynomial identities are interpreted through column coordinate vectors and determined by the matrix multiplication involving A and the $n \times n$ identity matrix. Each entry of the resulting matrix yields a polynomial equation in the structure constants, constructing a complete system of equations that represents algebraic conditions characterizing the targeted class of algebras. This method provides a constructive link between identity-based definitions and equation-based classification techniques.

Keywords: Matrix of structure constants; polynomial identities; basis vectors; identity conversion

Introduction

There are two primary approaches to the classification of algebra structures: the structural (basis-free, invariant) approach and the coordinate (basis-dependent, structure constants) approach. It is observed that the structural approach is difficult when dealing with general types of algebras. For example, Ariffin et al. (2020) utilized invariant functions method to classify low dimensional non-Lie filiform Leibniz algebras over Z_p. The approach can only facilitate classification of filiform algebras. Another approach is known as the coordinate-based (structure constants) approach, which is observed to be limited to classify low-dimensional algebras due to extensive computations required in solving such problem. This approach was utilised in the work of Goze and Remm (2011), where they provided a complete classification of all two-dimensional algebra over field K by analyzing the action of the linear group GL(2, K) parameterized by the structure constants.

Rakhimov et al. (2020) provided numerous polynomial identities for finite-dimensional algebras. A list of well-known single polynomial identities was given and the classification of all two-dimensional algebras with respect to these identities is given. This literature motivates this study to present a coordinate-based framework that converts polynomial identities into matrix form, whereby a solvable system of equations can be produced to solve algebraic classification problem.

Preliminaries

This section gives the preliminary knowledge to fully engage with this study.

Definition 1 (Ayupov et al., 2019). An Algebra A = **(*V*, ·)** over **F** is a vector space **V** over **F** with bilinear binary operation **·** : $V \times V \to V$ satisfies the following conditions

$$(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z);$$
$$z \cdot (\alpha x + \beta y) = \alpha(z \cdot x) + \beta(z \cdot y)$$

whenever $x, y, z \in V$ and $\alpha, \beta \in F$.

Let **A** be any n-dimensional algebra over F with multiplication • given by bilinear map (**x,y**) → x • y for all x,y ∊ **A**. If $e = \{e_1, e_2, ..., e_n\}$ is a basis for algebra **A** over **F**, then we can write

$$x = \sum_{i=1}^{n} e_i x_i = ex, \quad y = \sum_{j=1}^{n} e_j y_j = ey,$$

where $x = (x_1, x_2, ..., x_n)$, and $y = (y_1, y_2, ..., y_n)$ are column coordinate vectors of **x** and **y**, respectively.

Definition 2 (Ahmed et al., 2020) An algebra **A** is said to be

(i) commutative if $x \cdot y = y \cdot x$;
(ii) anti-commutative if $x \cdot y = -y \cdot x$;
(iii) associative if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(iv) anti-associative if $(x \cdot y) \cdot z = -x \cdot (y \cdot z)$;

for all **x**, **y, z** ∈ **A**.

An $n \times n$ matrix *A*

$$A = [\gamma_{1,1}^{1} \cdots \gamma_{1,n}^{1} \vdots \ddots \vdots \gamma_{1,1}^{n} \cdots \gamma_{n,1}^{n} \quad \gamma_{2,1}^{1} \cdots \gamma_{2,n}^{1} \vdots \ddots \vdots \gamma_{2,1}^{n} \cdots \gamma_{2,n}^{n} \quad \cdots \quad \gamma_{n,1}^{1} \cdots \gamma_{n,n}^{1} \vdots \ddots \vdots \gamma_{n,1}^{n} \cdots \gamma_{n,n}^{n}]$$

is said to be a Matrix of Structure Constant (shortly, MSC) of **A** on the basis $e = \{e_1, e_2, ..., e_n\}$ if its entries $\gamma_{i,j}^{k}$, for i, j, k = 1, 2, 3, ..., n, are defined as follows:

$$e_i \bullet e_j = \Upsilon_{i,j}^{1} + \Upsilon_{i,j}^{2} + \cdots + \Upsilon_{i,j}^{n} = \sum_{i,j,k=1}^{n} \gamma_{i,j}^{k} e_k$$

where $\gamma_{i,j}^{k}$ are the structure constants of **A** with respect to the basis **e**. Let for all x, y from algebra **A,** we have $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ and $y = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n$. Then

$$x \bullet y = eA(x \otimes y) \tag{1}$$

where $(x \otimes y) = (x_1y_1, x_2y_2, x_2y_1, x_2y_2, \ldots, x_ny_n)$. It is clear that $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are coordinate vectors of x and y, respectively.

Results and discussion

In this section, we provided the lemmas to identify commutative, anti-commutative, associative, and anti-associative algebras.

Lemma 1 An algebra **A** is commutative if and only if $A(x \otimes y - y \otimes x) = 0$, where A is an MSC of **A** and *x, y* are coordinate vectors of **x**, **y** ∈ **A**.

Proof. From Definition 2(i), we have $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. It implies that $\mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} = 0$.
Now, by equation (1), $\mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} = 0$ becomes

$$\boldsymbol{e}A(x \otimes y) - \boldsymbol{e}A\,(y \otimes x) = 0,$$

which leads to $[A(x \otimes y) - A\,(y \otimes x)] = 0$, and thus

$$A(x \otimes y) - A\,(y \otimes x) = 0.$$

Therefore, $A\,(x \otimes y - y \otimes x) = 0$. ■

Lemma 2 An algebra *A* is anti-commutative if and only if $A\,(x \otimes y + y \otimes x) = 0$, where A is an MSC of **A** and *x, y* are coordinate vectors of **x**, **y** ∈ **A**.

Proof. From Definition 2(ii), we have $\mathbf{x} \cdot \mathbf{y} = -\mathbf{y} \cdot \mathbf{x}$. This implies that $\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} = 0$.
Now, by equation (1), $\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} = 0$ becomes

$$\boldsymbol{e}A(x \otimes y) + \boldsymbol{e}A\,(y \otimes x) = 0$$

which leads to $[A(x \otimes y) + A\,(y \otimes x)] = 0$, and thus

$$A(x \otimes y) + A\,(y \otimes x) = 0.$$

Therefore, $A(x \otimes y + y \otimes x) = 0$. ■

Lemma 3 An algebra **A** is associative if and only if $A\ (A \otimes I) - A(I \otimes A) = 0$, where A is an MSC of **A** and I is a matrix identity of A.

Proof. From Definition 2(iii), $(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$ implies $(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} - \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = 0$. Now,

$$\begin{aligned}
(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} - \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) &= 0 \\
\boldsymbol{e}(\boldsymbol{e}A(x \otimes y) \otimes z) - \boldsymbol{e}A(x \otimes (\boldsymbol{e}A(y \otimes z))) &= 0; \\
\boldsymbol{e}\left[\left(\boldsymbol{e}A(x \otimes y) \otimes z\right) - \left(x \otimes \left(\boldsymbol{e}A(y \otimes z)\right)\right)\right] &= 0; \\
\left[\left(A(x \otimes y) \otimes z\right) - \left(x \otimes \left(A(y \otimes z)\right)\right)\right] &= 0; \text{(since } \boldsymbol{e} \neq 0) \\
\left[\left(A(x \otimes y) \otimes Iz\right) - \left(Ix \otimes \left(A(y \otimes z)\right)\right)\right] &= 0; \text{(allowing tensor operation)}
\end{aligned}$$

$$\begin{aligned}
[(A \otimes I)(x \otimes y) \otimes z) \ - (I \otimes A)\left((x \otimes (y \otimes z))\right] &= 0; \text{(by tensor product property)}\\
[(A \otimes I)\ (x \otimes y \otimes z)) \ - ((I \otimes A)\ (x \otimes y \otimes z))] &= 0;\\
[(A \otimes I) \ - (I \otimes A)]\ (x \otimes y \otimes z) &= 0.
\end{aligned}$$

Thus $(A \otimes I)$ $-A(I \otimes A)$ = 0 since $(x \otimes y \otimes z) \neq 0$. ■

Lemma 4 An algebra **A** is anti- associative if and only if *A (A ⊗ I) + A(I ⊗ A) = 0*, where *A* is an MSC of **A** and *I* is a matrix identity of *A*.

Proof. From Definition 2(iv), $(\mathbf{x} \cdot \mathbf{y}) \ \cdot \mathbf{z} = \ - \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$ implies $(\mathbf{x} \cdot \mathbf{y}) \ \cdot \mathbf{z} + \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = 0$. Now,

$$\begin{aligned}
(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} + \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) &= 0\\
\boldsymbol{e}(\boldsymbol{e}A(x \otimes y) \otimes z) + \boldsymbol{e}A\left(x \otimes \left(\boldsymbol{e}A(y \otimes z)\right)\right) &= 0\\
\boldsymbol{e}\left[(\boldsymbol{e}A(x \otimes y) \otimes z) + \left(x \otimes \left(\boldsymbol{e}A(y \otimes z)\right)\right)\right] &= 0\\
\left[(A(x \otimes y) \otimes z) + \left(x \otimes \left(A(y \otimes z)\right)\right)\right] &= 0; \text{(since } \boldsymbol{e} \ \neq \ 0)\\
\left[(A(x \otimes y) \otimes Iz) + \left(Ix \otimes \left(A(y \otimes z)\right)\right)\right] &= 0; \text{(allowing tensor operation)}\\
[(A \otimes I)(x \otimes y) \otimes z) + (I \otimes A)\left((x \otimes (y \otimes z))\right] &= 0; \text{(by tensor product property)}\\
[(A \otimes I)\ (x \otimes y \otimes z)) \ + ((I \otimes A)\ (x \otimes y \otimes z))] &= 0;\\
[(A \otimes I) \ + (I \otimes A)]\ (x \otimes y \otimes z) &= 0.
\end{aligned}$$

Hence $(A \otimes I) \ + A(I \otimes A) \ = \ 0$ since $(x \otimes y \otimes z) \neq 0$. ■

The following table shows the identity in MSC form for the identities of commutativity, anti-commutativity, associativity and anti-associativity.

Table 1: The Identities of selected algebra classes

Algebra Class	**Polynomial Identity**	**Identity in MSC form**
Commutative	$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$	$A(x \otimes y \ - \ y \otimes x) = 0$
Anti-commutative	$\mathbf{x} \cdot \mathbf{y} = -\mathbf{y} \cdot \mathbf{x}$	$A(x \otimes y \ + y \otimes x) = 0$
Associative	$(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$	$A(A \otimes I) - A(I \otimes A) = 0$
Anti-associative	$(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} = -\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$	$A(A \otimes I) + A(I \otimes A) = 0$

Conclusion

It is observed that the polynomial identities of algebras can be defined in the MSC form. The identity in MSC form allows matrix multiplication operation and each entry from the resulting matrix corresponds to one equation. Collectively, a solvable system of equations is obtained for each algebra class.

In the classification of finite-dimensional algebras, the structure constant approach, although general, quickly becomes computationally tedious as the dimension increases due to extensive coordinate expansions of polynomial identities. The matrix of structure constants (MSC) method reorganizes these coefficients into a linear map, allowing polynomial identities to be expressed as matrix equations. Although MSC does not reduce computational complexity in higher dimensions, it provides a systematic and transparent framework for identity conversion, isomorphism actions, and system generation using linear algebra.

Past research carried out by have presented some classification of finite-dimensional algebra structures over fields. Example given in the works of Kobayashi et al. (2021), Rakhimov (2024) and Asrorov et al. (2025). Besides that, the polynomial identities of finite-dimensional algebra in various classes have been given by Ahmed (2020). This study serves to present the framework in transforming the polynomial identities to MSC form, whereby a solvable system of equations can be generated for commutative and associative algebras. The same methods are applicable for all finite-dimensional algebras. Therefore, in future studies, one can explore the classification of various algebras over various field and vector space, utilizing MSC method. It is also good to conduct a study on the development of computational tools or algorithms to automate MSC- based classification for more complex or higher-dimensional algebras.

Acknowledgement

This research was funded by Universiti Putra Malaysia through GP-IPS 9789100.

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