Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Volume 34, 2026, page 7-11

Spectrum and Laplacian Spectrum of the Prime Power Order Cayley Graph for Cyclic Groups of Order pq

Hemappriya Shanmuganathan[a], Hazzirah Izzati Mat Hassim[a]*
[a]Department of Mathematical Sciences, Faculty of Science,
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia.
*Corresponding author: hazzirah@utm.my

Abstract
Spectral properties play a central role in the study of graphs, providing valuable insight into their structural characteristics. For a finite undirected graph Γ, the spectrum is defined as the set of eigenvalues of its adjacency matrix, while the Laplacian spectrum consists of the eigenvalues of its Laplacian matrix. A prime power order Cayley graph $Cay_{pp}(G, S^{(p_i^r)})$ is a graph associated with a finite group G, whose vertices correspond to the elements of G and $S^{(p_i^r)}$ is the subset of G consisting of non-identity elements with order p_i^r. Two distinct vertices u and v are adjacent if and only if $uv^{-1} \in S^{(p_i^r)}$. In this paper, a general construction of such graphs associated with cyclic groups of order pq, where p and q are distinct primes, is presented, and their spectrum and Laplacian spectrum are determined.
Keywords: Spectrum; Laplacian spectrum; Cayley graph; cyclic group

Introduction

Spectral graph theory is a branch of mathematics that explores the structural properties of graphs associated with matrices, particularly the adjacency and Laplacian matrices. The spectral properties of these matrices, such as characteristic polynomials, eigenvalues, and eigenvectors, provide valuable insights into graph structures and have applications in various domains, including network analysis, computer science, physics, and chemistry. In particular, the multiset of eigenvalues of the adjacency and Laplacian matrices, together with their multiplicities, defines the spectrum and Laplacian spectrum of a graph. These spectral properties have also been widely studied for graphs associated with algebraic structures, including groups.

Cayley graphs are a classical class of graphs constructed from groups and are widely used to study algebraic properties of groups defined by a chosen generating set [1]. For a finite group G and an inverse-closed subset $S \subseteq G \setminus \{e\}$, the Cayley graph, denoted by $Cay(G, S)$ is a simple undirected graph with vertex set G, where two vertices g and h are adjacent if and only if $gh^{-1} \in S$. In recent years, numerous studies have investigated new extensions and constructions of Cayley graphs on finite groups (see [2-4]). For instance, Tolue [5, 6] introduced the prime and composite order Cayley graphs, where the subset consists of all elements of prime and composite order, respectively, while Zulkarnain et al. [7] introduced the prime power Cayley graph based on the subset of all elements of prime power order. In 2024, Shanmuganathan [8] introduced the prime power order Cayley graph, defined on the subsets of elements with specific order. Furthermore, the spectral analysis of Cayley graphs has attracted considerable attention (see [9-11]). Ghorbani and Larki [12] investigated the spectrum of

Cayley graphs of certain groups of order $2pq$ and $3pq$. Afshari and Maghasedi [13] studied the eigenvalues of Cayley graphs of generalized dihedral groups. Recently, Árnadóttir and Godsil [14] explored the eigenvalues of integral normal Cayley graphs of finite groups of odd order.

In this paper, the prime power order Cayley graphs are constructed for cyclic groups of order pq, where p and q are distinct primes. By analyzing the structure of these graphs, their spectrum and Laplacian spectrum are determined.

Preliminaries

In this section, some basic concepts, definitions and preliminary results used in this study are presented.

All graphs in this paper are assumed to be finite, simple, and undirected. The following notation and terminology for graphs are stated as in [15].

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The number of vertices of Γ, called the order of Γ, is denoted by $|V(\Gamma)|$. Two vertices u and v in Γ are said to be adjacent, denoted by $u \sim v$, if they are connected by an edge, and $u \nsim v$ otherwise. A complete graph on n vertices, denoted by K_n, is a simple graph in which every pair of distinct vertices is joined by an edge. The disjoint union of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. In general, $m\Gamma$ denotes the disjoint union of m copies of Γ.

In the following, some results on the spectrum and Laplacian spectrum of certain graphs relevant to this study are provided.

Proposition 1 [16] The spectrum of the complete graph K_n is $Spec(K_n) = \{(n-1)^1, (-1)^{n-1}\}$, and its Laplacian spectrum is $L_{spec}(K_n) = \{0^1, n^{n-1}\}$.

Proposition 2 [16] Let Γ be a graph with connected components Γ_i for $1 \leq i \leq s$. The spectrum of Γ is the union of the spectra of Γ_i, with multiplicities added. The same holds for the Laplacian spectrum.

Proposition 3 [16] The multiplicity of 0 as a Laplacian eigenvalue of a graph Γ equals the number of connected components of Γ.

Next, the definition of the prime power order Cayley graph of a finite group G with respect to the subset $S^{(p_i^r)} \subseteq G \setminus \{e\}$ consisting elements of order p_i^r is given.

Definition 1 [8] Let G be a group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_i are primes and $\alpha_i \in \mathbb{N}$ for $i = 1,2,\dots,k$. Let $S^{(p_i^r)} = \{g \in G : |g| = p_i^r\}$ be an inverse-closed subset of G for $r = 1,2,\dots,\alpha_i$. The prime power order Cayley graph, denoted by $Cay_{pp}(G, S^{(p_i^r)})$, is a graph with $V\left(Cay_{pp}(G, S^{(p_i^r)})\right) = G$, and two distinct vertices g and h are adjacent if and only if $gh^{-1} \in S^{(p_i^r)}$.

Results and discussion

In this section, the general construction of the prime power order Cayley graph for a cyclic group G of order pq, where p and q are distinct primes, is presented. Then, the spectrum and Laplacian spectrum of these graphs are derived based on the general construction.

Let G be a cyclic group of order pq, generated by x, where p and q are distinct primes. By Definition 1, there are two subsets of G consisting of elements of specific prime power order,

(i) $S^{(p)} = \{g \in G : |g| = p\} = \{x^{iq} : 1 \leq i \leq p-1\}$,

(ii) $S^{(q)} = \{g \in G : |g| = q\} = \{x^{ip} : 1 \leq i \leq q-1\}$.

Thus, there are two prime power order Cayley graphs of G with respect to the subsets $S^{(p)}$ and $S^{(q)}$. First, the prime power order Cayley graph of G with respect to $S^{(p)}$, $Cay_{pp}(G, S^{(p)})$, is constructed using the following lemmas.

Lemma 1 Let G be a cyclic group of order pq, and let $S^{(p)} = \{g \in G : |g| = p\}$. For $1 \leq j \leq q$, let $V_j = \{x^{iq+(j-1)} : 0 \leq i \leq p-1\}$. Then, G is the disjoint union of the sets $V_1, V_2, \dots, V_q$.

Proof. Let $G = \langle x \rangle$ with $|x| = pq$. Then, $G = \{x^t : 0 \leq t \leq pq-1\}$. Let $t = iq + (j-1)$ for $0 \leq i \leq p-1$. Then, $x^t \in V_j$ for $1 \leq j \leq q$. Each V_j contains p distinct elements, and since there are q sets, their union contains pq distinct elements. Hence, $G = V_1 \cup V_2 \cup \cdots \cup V_q$. ∎

Lemma 2 Let G be a cyclic group of order pq, and let $S^{(p)} = \{g \in G : |g| = p\}$. For $1 \leq j \leq q$, let $V_j = \{x^{iq+(j-1)} : 0 \leq i \leq p-1\}$. Then, every pair of distinct vertices in V_j are adjacent.

Proof. Let $g, h \in V_j$ such that $g = x^{iq+(j-1)}$ and $h = x^{i'q+(j-1)}$ for $0 \leq i, i' \leq p-1$ and $i \neq i'$. Then, $gh^{-1} = x^{(i-i')q} \in S^{(p)}$. Therefore, $g \sim h$ for all $g \neq h \in V_j$. ∎

Lemma 3 Let G be a cyclic group of order pq, and let $S^{(p)} = \{g \in G : |g| = p\}$. For $1 \leq j, j' \leq q$ and $j \neq j'$, let $V_j = \{x^{iq+(j-1)} : 0 \leq i \leq p-1\}$ and $V_{j'} = \{x^{i'q+(j'-1)} : 0 \leq i' \leq p-1\}$. Then, every vertex in V_j are not adjacent to any vertex in $V_{j'}$.

Proof. Let $g \in V_j$ and $h \in V_{j'}$ for $j \neq j'$.

Case 1: If $i = i'$, then $gh^{-1} = x^{j-j'} \notin S^{(p)}$.
Case 2: If $i \neq i'$, then $gh^{-1} = x^{(i-i')q+(j-j')} \notin S^{(p)}$.

Therefore, $g \nsim h$ for every $g \in V_j$ and $h \in V_{j'}$. ∎

By using these lemmas, the following theorem for the prime power order Cayley graphs of G with respect to $S^{(k)}$, $Cay_{pp}(G, S^{(k)})$, where $k \in \{p, q\}$, is established.

Theorem 1 Let G be a cyclic group of order pq, where p and q are distinct primes. Let $S^{(k)} = \{g \in G : |g| = k\}$ for $k \in \{p, q\}$. The prime power order Cayley graphs of G with respect to $S^{(k)}$ are:

$$Cay_{pp}(G, S^{(k)}) = \begin{cases} qK_p & \text{if } k = p, \\ pK_q & \text{if } k = q. \end{cases}$$

Proof. Let G be a cyclic group of order pq, generated by x. Then, $G = \{e, x, x^2, x^3, \dots, x^{pq-1}\}$. For $k \in \{p, q\}$, let $S^{(k)}$ be a non-empty subset of G such that $S^{(k)} = \{g \in G : |g| = k\} = \{x^{ipq/k} : 1 \leq i \leq k-1\}$.

The structure of $Cay_{pp}(G, S^{(k)})$ is considered for the following cases:

Case 1: $k = p$. By Lemma 1, the elements of G are partitioned into q disjoint sets. Hence, by Definition 1, $V\left(Cay_{pp}\left(G, S^{(p)}\right)\right) = G = \bigcup_{j=1}^{q} V_j$. By Lemma 2, every pair of distinct vertices in V_j is adjacent, forming a complete graph K_p, while Lemma 3 implies that the vertices in distinct sets V_j and $V_{j'}$ for $j \neq j'$ are not adjacent. Therefore, $Cay_{pp}\left(G, S^{(p)}\right)$ is the disjoint union of q complete graphs, that is, $Cay_{pp}\left(G, S^{(p)}\right) = \underbrace{K_p \cup K_p \cup \cdots \cup K_p}_{q \text{ times}} = qK_p$.

Case 2: $k = q$. The proof follows a similar approach as in Case 1. ■

Next, the spectrum and Laplacian spectrum of the prime power order Cayley graphs associated with cyclic groups of order pq, are obtained, as stated in Proposition 4 and Proposition 5.

Proposition 4 Let G be a cyclic group of order pq, and let $S^{(k)} = \{g \in G: |g| = k\}$ for $k \in \{p, q\}$. Then, the spectrum of $Cay_{pp}\left(G, S^{(k)}\right)$ is

$$Spec\left(Cay_{pp}\left(G, S^{(k)}\right)\right) = \begin{cases} \{(p-1)^q, (-1)^{q(p-1)}\} & \text{if } k = p, \\ \{(q-1)^p, (-1)^{p(q-1)}\} & \text{if } k = q. \end{cases}$$

Proof. We compute the spectrum of $Cay_{pp}\left(G, S^{(p^k)}\right)$ in two cases.

Case 1: $k = p$. By Theorem 1, $Cay_{pp}\left(G, S^{(p)}\right) = qK_p$. By Proposition 1,

$$Spec\left(K_p\right) = \{(p-1)^1, (-1)^{p-1}\}.$$

Then by Proposition 2,

$$Spec\left(Cay_{pp}\left(G, S^{(p)}\right)\right) = \{(p-1)^q, (-1)^{q(p-1)}\}.$$

Case 2: $k = q$. The proof follows a similar approach as in Case 1. ■

Proposition 5 Let G be a cyclic group of order pq, and let $S^{(k)} = \{g \in G: |g| = k\}$ for $k \in \{p, q\}$. Then, the Laplacian spectrum of $Cay_{pp}\left(G, S^{(k)}\right)$ is

$$L_{spec}\left(Cay_{pp}\left(G, S^{(k)}\right)\right) = \begin{cases} \{0^q,\ p^{q(p-1)}\} & \text{if } k = p, \\ \{0^p,\ q^{p(q-1)}\} & \text{if } k = q. \end{cases}$$

Proof. We compute the Laplacian spectrum of $Cay_{pp}\left(G, S^{(p^k)}\right)$ in two cases.

Case 1: $k = p$. By Theorem 1, $Cay_{pp}\left(G, S^{(p)}\right) = qK_p$. By Proposition 1,

$$L_{spec}\left(K_p\right) = \{0^1,\ p^{p-1}\}.$$

Then by Proposition 2 and Proposition 3,

$$L_{spec}\left(Cay_{pp}\left(G, S^{(p)}\right)\right) = \{0^q,\ p^{q(p-1)}\}.$$

Case 2: $k = q$. The proof follows a similar approach as in Case 1. ■

Conclusion

In this paper, the prime power order Cayley graphs of the cyclic group G of order pq, where p and q are distinct primes, are constructed with respect to the subsets $S^{(p)}$ and $S^{(q)}$, and the resulting graphs are found to be isomorphic to qK_p and pK_q. Moreover, their spectrum are $\{(p-1)^q, (-1)^{q(p-1)}\}$ and $\{(q-1)^p, (-1)^{p(q-1)}\}$, while their Laplacian spectrum are $\{0^q, p^{q(p-1)}\}$ and $\{0^p, q^{p(q-1)}\}$, respectively.

Acknowledgement

The authors would like to acknowledge the support provided by Universiti Teknologi Malaysia under the Potential Academic Staff Grant (Q.J130000.2754.04K27).

References

[1] Cayley, P. 1878. Desiderata and Suggestions: No. 2. The theory of groups: graphical representation. American Journal of Mathematics. 1(2): 174-176.
[2] Zulkarnain, A., Sarmin, N. H., Mat Hassim, H. I., and Erfanian, A. 2020. The direct product of p_i-Cayley graph for Alt (4) and Sym (4). AIP Conference Proceedings.
[3] Neamah, A. A, Erfanian, A., and Majeed A. H. 2022. On a generalized Cayley graph of column matrices of elements of a finite group. MATHEMATICA. 64(87)(2): 247-254.
[4] Susanti, Y., and Erfanian, A. 2024. Prime square order Cayley graph of cyclic groups. Asian-European Journal of Mathematics. 17(2).
[5] Tolue, B. 2015. The prime order Cayley graph. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics. 77(3): 207-218.
[6] Tolue, B. 2019. Some graph parameters on the composite order Cayley graph. Caspian Journal of Mathematical Sciences. 8(1): 10-17.
[7] Zulkarnain, A., Sarmin, N. H., Mat Hassim, H. I., and Erfanian, A. 2022. Prime power Cayley graph of a cyclic group. Proceedings of Science and Mathematics. 7: 25–28.
[8] Shanmuganathan, H. 2024. *The prime power order Cayley graph and the union prime power order Cayley graph of certain cyclic groups and their topological indices* [master's thesis, Universiti Teknologi Malaysia].
[9] Krakovski, R., and Mohar, B. 2012. Spectrum of Cayley graphs on the symmetric group generated by transpositions. Linear Algebra and its Applications. 437(3): 1033-1039.
[10] Ghorbani, M., and Songhori, M. 2020. On the spectrum of Cayley graphs. Algebra and Discrete Mathematics. 30: 194-206.
[11] Saha, J. P. 2023. On the spectral gap of Cayley graphs. arXiv preprint arXiv:231206604.
[12] Ghorbani, M., and Larki, F. N. 2018. On the spectrum of finite Cayley graphs. Journal of Discrete Mathematical Sciences and Cryptography. 21(1): 83-112.
[13] Afshari, F., and Maghasedi, M. 2019. On the eigenvalues of Cayley graphs on generalized dihedral groups. Algebraic Structures and Their Applications. 6(2): 39-45.
[14] Árnadóttir, A. S., and Godsil, C. 2025. A note on eigenvalues of Cayley graphs. Linear Algebra and its Applications. 721: 240-249.
[15] Gross, J. L., Yellen, J., and Zhang, P. 2014. *Handbook of Graph Theory*, 2nd ed. CRC Press.
[16] Brouwer, A. E., and Haemers, W. H. 2012. *Spectra of Graphs*. Springer.